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The biinvariant diagonal class for Hamiltonian torus actions

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Abstract

Suppose that an algebraic torus G acts algebraically on a projective manifold X with generically trivial stabilizers. Then the Zariski closure of the set of pairs $\{(x, y) \in X \times X \mid y = gx \text{ for some } g \in G\}$ defines a nonzero equivariant cohomology class $[\Delta_G] \in H^*_{G\times G}(X \times X)$. We give an analogue of this construction in the case where X is a compact symplectic manifold endowed with a Hamiltonian action of a torus, whose complexification plays the role of G. We also prove that the Kirwan map sends the class $[\Delta_G]$ to the class of the diagonal in each symplectic quotient. This allows to define a canonical right inverse of the Kirwan map.

Keywords: Hamiltonian torus actions; Kirwan map

Contents

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1. Introduction

1.1. The purpose of this paper is to give a symplectic version of the following construction in algebraic geometry. Let X be a smooth projective scheme over $\mathbb{C}$ of complex dimension n endowed with an algebraic action of an algebraic group G with generically trivial isotropy groups. Consider the following subset of $X \times X$:

$$\Delta_G = \{(x, y) \in X \times X \mid \text{there is some } g \in G \text{ such that } y = gx\}.$$

The set Δ_G is a constructible subset of $X \times X$ because it is the image of the algebraic map $f : X \times G \to X \times X$ defined as $f(x, g) = (x, gx)$. This implies that the inclusion of Δ_G in its Zariski closure $\overline{\Delta}_G \subset X \times X$ has dense image with respect to the analytic topology (see Ex. 3.18 and 3.19 in Chapter II of [5], or §4.4 in [7]) and hence that the dimension of $\overline{\Delta}_G$ is equal to that of Δ_G. Since the isotropy groups of the action are generically trivial, the dimension of Δ_G is equal to $n + \dim G$ (see for example Ex. 3.22 in Chapter II of [5]), so via the cycle map and Poincaré duality $\overline{\Delta}_G$ defines a nonzero cohomology class $[\Delta_G]_0 \in H^{2(n-\dim G)}(X \times X; \mathbb{Z})$. The set $\overline{\Delta}_G$ is invariant under the product action of $G \times G$ on $X \times X$ (in contrast with the usual diagonal which is only invariant under the diagonal action of G on $X \times X$). Using algebraic finite-dimensional approximations of the classifying space BG together with a stabilization argument, one can apply the previous reasoning to define a nonzero equivariant cohomology class

$$[\Delta_G] \in H^{2(n-\dim G)}_{G\times G}(X \times X; \mathbb{Z}).$$

We call $[\Delta_G]$ the biinvariant diagonal class.

1.2. In this paper we generalize the previous construction to symplectic geometry when G is the complexification of a compact torus T. We also prove some properties of $[\Delta_G]$, which we construct over the rationals and not over the integers. Let (X, ω) be a compact connected symplectic manifold of real dimension $2n$, endowed with an effective Hamiltonian action of T, which for the moment we take to be S^1. Denote by $\mu : X \to (\mathbf{i}\mathbb{R})^*$ the moment map and define the function $h : X \to \mathbb{R}$ as $h = \langle \mu, \mathbf{i} \rangle$. Fix an invariant Riemannian metric on X of the form $\omega(\cdot, I\cdot)$, where I is an invariant almost complex structure on X. Let $\xi_t : X \to X$ be the downward gradient flow of h, defined by the conditions that ξ_0 is the identity and $\xi_t' = -\xi_t^* \nabla h$. Define

$$\Delta_{\mathbb{C}^*} = \{(x, y) \in X \times X \mid \text{there is some } t \in \mathbb{R} \text{ and } \theta \in S^1 \text{ such that } y = \theta \cdot \xi_t(x)\}.$$

When $[\omega/2\pi] \in H^2(X; \mathbb{Z})$ and I is integrable then X is projective by Kodaira's theorem, the action of S^1 extends to an algebraic action of $\mathbb{C}^*$, and for any $z \in \mathbb{C}^*$ and $x \in X$ we have $z \cdot x = \theta \cdot \xi_t(x)$, where $\theta = z/|z|$ and $t = \ln |z|$. Hence this definition of $\Delta_{\mathbb{C}^*}$ generalizes the one in Section 1.1 and consequently, denoting by $\overline{\Delta}_{\mathbb{C}^*}$ the closure of $\Delta_{\mathbb{C}^*}$ in the standard topology of $X \times X$, the complement $\overline{\Delta}_{\mathbb{C}^*} \setminus \Delta_{\mathbb{C}^*}$ has a natural structure of stratified space of real dimension $\dim_{\mathbb{R}} \Delta_{\mathbb{C}^*} - 2$.

Unlike in the algebraic case, in general there is no reason to expect that $\overline{\Delta}_{\mathbb{C}^*} \setminus \Delta_{\mathbb{C}^*}$ has smaller dimension than $\Delta_{\mathbb{C}^*}$ in any sense which would allow $\overline{\Delta}_{\mathbb{C}^*}$ to define a homology class of real dimension $2n + 2$. However, using multivalued perturbations of the gradient flow equation (see for example §5.2 in [16], or Section 2.2 in this paper for the notion of multivalued perturbation) we can define a nonzero rational cohomology class

$$[\Delta_{\mathbb{C}^*}] \in H^{2n-2}_{S^1\times S^1}(X \times X),$$

which is morally the equivariant Poincaré dual of the class represented by $\Delta_{\mathbb{C}^*}$, and which coincides in the algebraic case with the class defined in Section 1.1. (Here and in the rest of the paper we omit the coefficients in (co)homology groups, which are always assumed to be $\mathbb{Q}$.)

The idea of considering multivalued perturbations for the gradient line equation to achieve simultaneously equivariance and transversality has been well known to experts for some time. For example, a sketch of this technique is explained in Lemma 4.7 of [12], where it is applied to the definition of the cohomology classes represented by stable and unstable manifolds. However, as far as we know a detailed exposition of this construction applied to the gradient line equation does not exist in the literature, and this was one of the motivations for writing this paper. Note that, in contrast, full details have been given of the technique of multivalued perturbations applied to much more involved geometric problems, such as the construction of Gromov–Witten invariants or Floer homology [3,10,15,16].

1.3. The main result of this paper is, besides the definition of $[\Delta_{\mathbb{C}^*}]$, the computation of its image under the diagonal Kirwan map. Recall that if m is a regular value of h the symplectic quotient (or reduced space, or Marsden–Weinstein quotient) of X at m is

$$Y_m = h^{-1}(m)/S^1.$$

The Kirwan map is the morphism of rings

$$\kappa_m : H^*_{S^1}(X) \to H^*(Y_m)$$

defined as the composition of the restriction map $H^*_{S^1}(X) \to H^*_{S^1}(h^{-1}(m))$ with the Cartan isomorphism $H^*_{S^1}(h^{-1}(m)) \simeq H^*(Y_m)$, which exists because the action of S^1 on $h^{-1}(m)$ has finite stabilizers. Now $(X \times X)_{S^1\times S^1}$ can be identified with $X_{S^1} \times X_{S^1}$, so Künneth's formula gives an isomorphism

$$H^*_{S^1\times S^1}(X \times X) \simeq \bigoplus H^*_{S^1}(X) \otimes H^*_{S^1}(X). \tag{1.1}$$

Denote by $\kappa^2_m : H^*_{S^1\times S^1}(X \times X) \to H^*(Y_m \times Y_m)$ the Kirwan map for the quotient of $X \times X$ at $(m, m) \in \mathbb{R}^2$. Let $[\Delta_m] \in H^*(Y_m \times Y_m)$ denote the Poincaré dual of the diagonal class (Poincaré duality holds on Y_m with rational coefficients because Y_m is an orbifold). We then have:

Theorem 1.1. *For each regular value $m \in \mathbb{R}$ of h we have $\kappa^2_m([\Delta_{\mathbb{C}^*}]) = [\Delta_m]$.*

1.4. A right inverse of the Kirwan map at regular quotients

Let $m \in \mathbb{R}$ be a regular value of the moment map. We deduce a number of consequences of Theorem 1.1 by looking at $(\kappa_m \otimes \mathrm{Id})[\Delta_{\mathbb{C}^*}]$ as a correspondence in the sense of intersection theory. More precisely, using Poincaré duality $PD : H^k(Y_m) \to H^{2n-2-k}(Y_m)^*$, we can write

$$(PD \otimes \mathrm{Id}) \circ (\kappa_m \otimes \mathrm{Id})[\Delta_{\mathbb{C}^*}] \in \bigoplus_{p+q=2n-2} H^{2n-2-p}(Y_m)^* \otimes H^q_{S^1}(X) = \bigoplus_q H^q(Y_m)^* \otimes H^q_{S^1}(X).$$

Hence $(PD \otimes \mathrm{Id}) \circ (\kappa_m \otimes \mathrm{Id})[\Delta_{\mathbb{C}^*}]$ gives rise to a degree preserving linear map

$$l_m : H^*(Y_m) \to H^*_{S^1}(X).$$

Using Theorem 1.1 we prove the following.

Corollary 1.2. *The map l_m is a right inverse of the Kirwan map, i.e., the composition $\kappa_m \circ l_m$ is the identity on $H^*(Y_m)$. In particular the Kirwan map κ_m is surjective.*

The map l_m is not in general morphisms of rings (see Section 5.4 for an example).

To state the next corollary we need to introduce some notation. The Poincaré dual of the class of the diagonal $[\Delta_m] \in H^*(Y_m \times Y_m)$ gives rise to a nondegenerate quadratic Q_m form on the homology $H_*(Y_m)$ defined as $Q_m(a,b) = \langle a \otimes b, [\Delta_m]\rangle$ for any $a, b \in H_*(Y_m)$ (this is the usual intersection product in homology). We can define similarly a quadratic form $Q_{\mathbb{C}}$ on the equivariant homology $H^{S^1}_*(X)$ by setting $Q_{\mathbb{C}}(\alpha,\beta) = \langle \alpha \otimes \beta, [\Delta_{\mathbb{C}^*}]\rangle$ for any $\alpha, \beta \in H^{S^1}_*(X)$. Theorem 1.1 implies the following.

Corollary 1.3. *Let m be any regular value of the moment map and let $\kappa^*_m : H_*(Y_m) \to H^{S^1}_*(X)$ be the dual of the Kirwan map. For any classes $a, b \in H_*(Y_m)$ we have*

$$Q_{\mathbb{C}}\big(\kappa^*_m(a), \kappa^*_m(b)\big) = Q_m(a,b).$$

Note that the quadratic form $Q_{\mathbb{C}}$ is always degenerate.

1.5. Modified product in equivariant cohomology

A way to encode the map l_m is in terms of an associative ring structure on the equivariant cohomology of X which is different from the usual one. Given classes $\alpha, \beta \in H^*_{S^1}(X)$ we define

$$\alpha \cup_m \beta = l_m\big(\kappa_m(\alpha) \cup \kappa_m(\beta)\big).$$

The product $\cup_m$ is associative because l_m is a right inverse of κ_m. In a sense, this is nothing but the usual product in the cohomology of the symplectic quotient transported via l_m to the equivariant cohomology. What makes this construction interesting is the possibility to define associative deformations of $\cup_m$ in terms of the so-called Hamiltonian Gromov–Witten invariants counting twisted holomorphic maps from $\mathbb{CP}^1$ to X, similarly to how the quantum product is defined (see [13,14] and the references therein).

1.6. The case of singular quotients

When m is a critical value of h the Kirwan map cannot be defined as in the case of regular values, since the cohomologies $H^*_{S^1}(h^{-1}(m))$ and $H^*(Y_m)$ need no longer be isomorphic. In this situation, the quotient Y_m being a singular stratified space, it is more natural to consider the (middle perversity) intersection cohomology $IH^*(Y_m)$ rather than singular cohomology. Lerman and Tolman have shown in [9] how to relate the equivariant cohomology of X to the intersection cohomology $IH^*(Y_m)$: assuming that m belongs to the interior of $h(X)$ (otherwise $h^{-1}(m)$ is a connected component of the fixed point set), they construct:

- an S^1-invariant Bott–Morse function $h' : X \to \mathbb{R}$, which is a slight perturbation of h, such that m is a regular value of h' and the action of S^1 on $h'^{-1}(m)$ has finite stabilizers;
- a map $f : h'^{-1}(m)/S^1 \to h^{-1}(m)/S^1$ which is a small resolution and hence induces an isomorphism f_H: $IH^*(h^{-1}(m)/S^1) \simeq H^*(h'^{-1}(m)/S^1)$ preserving the intersection pairing.

Let $Y'_m := h'^{-1}(m)/S^1$ and let us denote by $\kappa'_m : H^*_{S^1}(X) \to H^*(Y'_m)$ the composition of the restriction to $h'^{-1}(m)$ with the Cartan isomorphism $H^*_{S^1}(h'^{-1}(m)) \to H^*(Y_m)$. It seems natural to call the composition $\kappa_m = f_H^{-1} \circ \kappa'_m$ *the Kirwan map* for the singular quotient $Y_m = h^{-1}(m)/S^1$. Kiem and Woolf give in [8] a definition of Kirwan maps at singular quotients for Hamiltonian actions of compact connected Lie groups. Presumably the map κ_m defined above can be obtained using their technique (but note that the Kirwan maps constructed in [8] are not canonical in general, whereas κ_m is canonical). Let us denote by $PD : IH^k(Y_m) \to IH^{2n-2-k}(Y_m)^*$ the Poincaré duality map.

Theorem 1.4. *The element $(PD \otimes \mathrm{Id}) \circ (\kappa_m \otimes \mathrm{Id})[\Delta_{\mathbb{C}^*}] \in \bigoplus_q IH^q(Y_m)^* \otimes H^q_{S^1}(X)$ corresponds to a degree preserving map $l_m : IH^*(Y_m) \to H^*_{S^1}(X)$ which is a right inverse of the Kirwan map, i.e., $\kappa_m \circ l_m$ is the identity in $IH^*(Y_m)$.*

1.7. Actions of any compact torus

Now suppose that X supports a Hamiltonian action of a compact torus T with Lie algebra $\mathfrak{t}$ and moment map $\mu : X \to \mathfrak{t}^*$. Take a basis $u_1, \dots, u_q$ of $\mathfrak{t}$ and consider continuous curves in X which are piecewise gradient lines for $\langle \mu, u_j \rangle$ (see Section 6 for details). Considering multivalued perturbations of the gradient line equations which are invariant under the action of a generic subgroup $S^1 \simeq T_0 \subset T$, one obtains a well-defined cohomology class, which is independent of the basis $\{u_i\}$ (but maybe depends on the choice of T_0). One can then prove the following.

Theorem 1.5. *Let q be the dimension of T. Let $S^1 \simeq T_0 \subset T$ be a subgroup such that the T_0-fixed point set coincides with the T-fixed point set. There is a cohomology class*

$$\left[\Delta_{\mathbb{C}^*}^{T_0,T}\right] \in H^{2n-2q}_{T \times T}(X \times X)$$

such that, for each regular value $m \in \mathfrak{t}^*$, $\kappa_m^2([\Delta_{\mathbb{C}^*}^{T_0,T}]) = [\Delta_m]$.

Similarly as in the case of S^1, the cohomology classes $[\Delta_{\mathbb{C}}^{T_0,T}]$ give rise to right inverses of the Kirwan map

$$l_m^{T_0,T} : H^*(Y_m) \to H_T^*(X).$$

1.8. Some remarks and questions

If $T = S^1$ acts on X quasi-freely (i.e., all isotropy groups are either trivial or the whole circle) then both $[\Delta_{\mathbb{C}^*}]$ and the right inverse l_m can be defined over the integers (using that the symplectic quotients are smooth and hence that Poincaré duality holds over the integers). Moreover, one can perturb the gradient flow equation using standard perturbations (i.e., not multivalued), hence everything is much easier. An immediate corollary is that if $H_T^*(X;\mathbb{Z})$ is torsion free then the cohomology of all symplectic quotients is torsion free. This is a particular case of Theorem 5 in [17], since for quasi-free actions of S^1 the group $H_T^*(F;\mathbb{Z})$ is torsion free if and only if $H_T^*(X;\mathbb{Z})$ is torsion free (this can be proved using the fact that, the action being quasi-free, the moment map is a perfect Bott–Morse function over any finite field).

An obvious question is whether the results in this paper extend to other situations in which the Kirwan map is known to be surjective: notably, the case of compact nonabelian groups (already considered by Kirwan) and the case of loop group actions, studied recently by Bott, Tolman and Weitsman in [2]. More generally, given any Hamiltonian action of a group G on a symplectic manifold X, one would like to understand the set of cohomology classes $[\Delta_{G^{\mathbb{C}}}] \in H^*_{G\times G}(X \times X)$ such that for each regular value $\alpha \in \mathfrak{g}^*$ of the moment map μ, denoting by $Y_\alpha = \mu^{-1}(\alpha)/G$ the symplectic reduction, one has $\kappa_\alpha^2([\Delta_{G^{\mathbb{C}}}]) = [\Delta_\alpha]$, where $[\Delta_\alpha] \in H^*(Y_\alpha \times Y_\alpha)$ is the diagonal class. A particular case of this question is whether the classes $[\Delta_{\mathbb{C}^*}^{T_0,T}]$ constructed in Theorem 1.5 depend on the choice of T_0. Another question is whether the class $[\Delta_G]$ can be defined over the integers, as is the case in the algebraic situation described in Section 1.1.

1.9. Contents of the paper

We now describe the contents of the remaining sections. In Section 2 we define the perturbed gradient segments which will be used to define the biinvariant diagonal. The biinvariant diagonal is defined, modulo Theorems 3.1 and 3.3, in Section 3. The proofs of Theorems 3.1 and 3.3 are given in Section 4. In Section 5 we prove Theorem 1.1, Corollary 1.2 and Theorem 1.4. Finally, in Section 6 we consider the case of higher-dimensional compact tori and sketch the proof of Theorem 1.5.

2. Perturbed gradient segments

Recall that I denotes an S^1-invariant almost complex structure on X which is compatible with ω, and that we consider on X the Riemannian metric $\omega(\cdot, I\cdot)$. Let $\mathcal{X}$ be the vector field on X generated by the infinitesimal action of $\mathbf{i} \in \mathbf{i}\mathbb{R} \simeq \operatorname{Lie} S^1$. The function h satisfies $dh = \iota_{\mathcal{X}}\omega$, so its gradient is $\nabla h = I\mathcal{X}$. In order to define the cohomology class $[\Delta_{\mathbb{C}^*}]$ we consider generic S^1-invariant perturbations of the downward gradient equation $\gamma' = -I\mathcal{X}(\gamma)$. The possible presence of finite isotropy groups forces us to consider multivalued perturbations in order to preserve S^1-invariance, which is crucial to bound the dimension of $\overline{\Delta}_{\mathbb{C}^*} \setminus \Delta_{\mathbb{C}^*}$ (see Lemma 4.3).

2.1. ϵ-perturbed gradient segments and some lemmata

Let $c_1 < \cdots < c_r \in \mathbb{R}$ be the critical values of h. Since the moment map is locally constant on the fixed point set $F \subset X$ and F coincides with the set of critical points of h, we have $h(F) = \{c_1, \ldots, c_m\}$. Choose real numbers $a_1, \ldots, a_{m-1}$ satisfying

$$c_1 < a_1 < c_2 < \cdots < c_{m-1} < a_{m-1} < c_m$$

and define $Z \subset X$ to be the union of the submanifolds $h^{-1}(a_1), \ldots, h^{-1}(a_{m-1})$. Take a number $\beta > 0$ satisfying $c_i + \beta < a_i < c_{i+1} - \beta$ for every i. Define

$$Z_i = h^{-1}\big([a_i - \beta, a_i + \beta]\big)$$

for each i and let Z' be the union $Z_1 \cup \cdots \cup Z_m$. Then the intersection $F \cap Z'$ is empty.

Definition 2.1. Let $A \subset \mathbb{R}$ be an interval and let ϵ be a positive number. A smooth map $\gamma : A \to X$ is called an *ϵ-perturbed gradient segment* if:

(1) $\gamma'(t) = -I\mathcal{X}(\gamma(t))$ whenever $\gamma(t) \notin Z'$ and
(2) there exists a tangent vector field $\mathcal{V}$ defined in an open neighborhood of the closure of $\gamma(A) \subset X$ satisfying $\gamma' = \mathcal{V}_\gamma$, $|\mathcal{V} + I\mathcal{X}|_{C^0} < \epsilon$ and $|\nabla(\mathcal{V} + I\mathcal{X})|_{C^0} < \epsilon$.

Unless otherwise specified, the domain of an ϵ-perturbed gradient segment will always be assumed to be an interval of $\mathbb{R}$. Define the following quantities:

$$M = \sup_{Z'} |I\mathcal{X}| \quad \text{and} \quad m = \inf_{Z'} |I\mathcal{X}|.$$

We will also always assume that $\epsilon \leqslant m/2$. This implies that if $\gamma : A \to X$ is an ϵ-perturbed gradient segment then $(h \circ \gamma)' < 0$, so γ is an embedding and the closure of $\gamma(A)$ is an embedded simply connected curve.

Lemma 2.2. *Suppose that $\epsilon < m^2$. Let $\gamma : A \to X$ be an ϵ-perturbed gradient segment and assume that $B = \gamma^{-1}(Z_i)$ is nonempty. Then B is connected and*

$$\frac{\operatorname{length}\gamma(B)}{M + \epsilon} \leqslant |B| \leqslant \frac{\sup h(\gamma(B)) - \inf h(\gamma(B))}{m(m - \epsilon)}. \tag{2.1}$$

Proof. We first estimate for any $\gamma(t) \in Z'$, using $\nabla h = I\mathcal{X}$ and Cauchy–Schwartz,

$$(h \circ \gamma)'(t) = \big\langle \gamma'(t), \nabla h \big\rangle = -\langle I\mathcal{X}, I\mathcal{X}\rangle + \big\langle \gamma'(t) + I\mathcal{X}, I\mathcal{X}\big\rangle \leqslant -m^2 + \epsilon m.$$

We prove that B is connected by contradiction. Suppose that $t_0, t_1 \in B$ but that $t_0 < \tau < t_1$ satisfies $\gamma(\tau) \notin Z_i$. Then either $h(\tau) > a_i + \beta$ or $h(\tau) < a_i - \beta$. In the first case there must exist some $t \in [\tau, t_1]$ such that $(h \circ \gamma)(t) = a_i + \beta$ and $(h \circ \gamma)'(t) \geqslant 0$, which by the estimate above contradicts our assumption $\epsilon < m$; the second case leads to a contradiction in the same way. Once we know that B is connected, integrating the inequality $(h \circ \gamma)'(t) \leqslant -m^2 + \epsilon m$ along B we get the second inequality in (2.1). To get the first inequality in (2.1) we estimate for any $\gamma(t) \in Z'$, similarly as before, $|\gamma'(t)| \leqslant M + \epsilon$, and then we integrate along B. $\square$

For any $z \in Z$ and any real number $\delta > 0$ we define the following set:

$$S(z, \delta) = \{\exp_z v \mid v \in T_z X,\ v \text{ is perpendicular to } \mathcal{X} \text{ and } |v| \leqslant \delta\}.$$

Let also $R(z, \delta) = S^1 \cdot S(z, \delta)$. If δ is smaller than the injectivity radius then $S(z, \delta)$ is a slice of the S^1 action at z. Also, if δ is small enough then $R(z, \delta)$ can be identified with a solid torus (i.e., the product of a circle and a closed ball) contained in Z'. The generalized Gauss lemma for submanifolds (see e.g. Lemma 2.11 in [4]) implies that, given $x \in X$ and $z \in Z$,

$$d(x, S^1 \cdot z) \text{ is small enough} \quad \Longrightarrow \quad d(x, S^1 \cdot z) = \inf\{\delta \mid x \in R(z, \delta)\}. \tag{2.2}$$

Lemma 2.3. *There exist positive numbers ϵ, δ_0 with the following property. Suppose that $\gamma : A \to X$ is an ϵ-perturbed gradient segment. Let $z \in Z$ and define, for any $t \in A$, $f(t) = \inf\{\delta^2 \mid \gamma(t) \in R(z, \delta)\}$. If $f(t) \leqslant \delta_0^2$, then $f''(t) \geqslant 1$.*

Proof. Take some $z \in Z$ and let $S \subset T_z X$ be the linear span of $\mathcal{X}_z$ and $I\mathcal{X}_z$. Let $U \subset S^\perp$ be a small neighborhood of 0. Choose some small $\eta > 0$ and let $V = (-\eta, \eta) \times (-\eta, \eta) \times U$. The map $\iota : V \to X$ defined as $\iota(t, \theta, u) = \xi_t(e^{\mathbf{i}\theta} \cdot \exp_z u)$ is an embedding and $\iota(V)$ is a neighborhood of z in X (recall that ξ_t is the downward gradient flow of h). We can identify $O = \{0\} \times (-\eta, \eta) \times \{0\} \subset V$ with $\iota^{-1}(S^1 \cdot z)$. Consider the Riemannian metric on V defined by $dg^2 = dt^2 + d\theta^2 + du^2$, where du^2 is the restriction to $S^\perp$ of the Euclidean pairing in $T_z X$, and let d_0 be the distance in V induced by dg^2. The integral curves of $d\iota^{-1}(I\mathcal{X})$ on V are given by $\gamma(t) := (t, \theta_0, u_0)$ for some constant $(\theta_0, u_0) \in (-\eta, \eta) \times U$, so the function $f_0(t) := d_0(\gamma(t), O)^2$ satisfies $f_0''(t) = 2$. Let d be the distance in V induced by the distance in X via the inclusion ι. If η and ϵ are small enough and $\mathcal{V}$ satisfies the hypothesis of Definition 2.1, then for any integral curve $\gamma_\mathcal{V}$ of $\mathcal{V}$ the function $f_\mathcal{V} = d(\gamma_\mathcal{V}, O)^2$ satisfies $f_\mathcal{V}'' \geqslant 1$. By (2.2) if f is small enough then $f = f_\mathcal{V}$. $\square$

Lemma 2.4. *Let $\epsilon = m/2$. For any $\delta_0 > 0$ there is some $0 < \delta_1 < \delta_0$ such that if $\gamma : A \to X$ is an ϵ-gradient segment and $z \in Z$ then $\gamma(\mathrm{Hull}(\gamma^{-1} R(z, \delta_1))) \subset R(z, \delta_0)$, where* Hull *denotes the convex hull.*

Proof. Take a positive $\delta_2 < \delta_0$ such that for any $z \in Z$ we have $R(z, \delta_2) \subset Z'$. Let d be the infimum for all points $z \in Z$ of the distance between the boundaries $\partial R(z, \delta_2/2)$ and $\partial R(z, \delta_2)$. If δ_2 has been chosen small enough, we have $d > 0$. Choose a positive $\delta_1 < \delta_2/2$ in such a way that for any $z \in Z$ we have

$$\sup_{R(z,\delta_1)} h - \inf_{R(z,\delta_1)} h < dm^2/2M. \tag{2.3}$$

We prove that δ_1 satisfies the requirement of the lemma, even replacing δ_0 by δ_2. Suppose that $\gamma : A \to X$ is an ϵ-gradient segment, and that for some $z \in Z$ there exist elements $\tau < \tau' < \tau''$ of A such that $\gamma(\tau), \gamma(\tau'') \in R(z, \delta_1)$ but $\gamma(\tau') \notin R(z, \delta_2)$. Let $B = [\tau, \tau'']$. By the triangle inequality length$(\gamma(B)) \geqslant 2d$. Combining both inequalities in (2.1) we have $h(\gamma(\tau)) - h(\gamma(\tau'')) \geqslant dm^2/2M$, contradicting (2.3). This proves the lemma. $\square$

Lemma 2.5. *There exist positive numbers ϵ, δ with the following property. Suppose that $\gamma : A \to X$ is an ϵ-perturbed gradient segment. Let $z \in Z$. Then*

(1) *The preimage $\gamma^{-1}R(z, \delta) \subset A$ is connected.*
(2) *If $\emptyset \neq \gamma(A) \cap R(z, \delta) \subset \partial R(z, \delta)$ then $\gamma^{-1}R(z, \delta)$ consists of a unique point.*

Proof. Let δ_0 be given by Lemma 2.3 and let δ_1 be the corresponding value given by Lemma 2.4. Let ϵ be less than the ϵ's in both lemmata and let $\delta = \delta_1$. If $\gamma : A \to X$ is an ϵ-perturbed gradient segment and $z \in Z$ then by Lemma 2.4 the convex hull B of $\gamma^{-1}R(z, \delta)$ satisfies $\gamma(B) \subset R(z, \delta_0)$. It follows that the function $f : B \to \mathbb{R}$ defined in Lemma 2.3 is convex, and hence $\gamma^{-1}R(z, \delta) \cap B$ is connected. Hence, $B = \gamma^{-1}R(z, \delta)$ and so the latter is connected. This proves (1), and (2) follows similarly. □

Lemma 2.6. *Suppose that $\gamma : A \to X$ is an ϵ-perturbed gradient segment and that A is a closed interval. Let $\mathcal{V}$ be the vector field defined in a neighborhood of $\gamma(A)$ as given by Definition* 2.1. *Then the following is true.*

(1) *One can take open neighborhoods $H \subset \mathbb{R}$ (respectively $O \subset X$) of $h(\gamma(\sup A))$ (respectively $\gamma(\inf A)$) such that for any $\lambda \in H$ and any $x \in O$ there is a unique integral curve $\gamma_{\lambda,x} : A_{\lambda,x} \to X$ of $\mathcal{V}$ such that $h(\gamma_{\lambda,x}(\sup A_{\lambda,x})) = \lambda$ and $\gamma_{\lambda,x}(\inf A) = x$.*

(2) *Take some point $z \in Z$ and define, for small enough $\delta > 0$, the following sets*:

$$\Sigma_\delta = \left\{(\lambda, x) \in H \times O \;\middle|\; \gamma_{\lambda,x}^{-1}R(z, \delta) \text{ consists of a unique point}\right\},$$

$$\Sigma_{\delta,1} = \left\{(\lambda, x) \in \Sigma_\delta \;\middle|\; \gamma_{\lambda,x}^{-1}R(z, \delta) \text{ belongs to the interior of } A_{\lambda,x}\right\},$$

$$\Sigma_{\delta,2} = \left\{(\lambda, x) \in \Sigma_\delta \;\middle|\; t = \gamma_{\lambda,x}^{-1}R(z, \delta) \in \partial A_{\lambda,x} \text{ and } \gamma_{\lambda,x}'(t) \text{ is not tangent to } \partial R(z, \delta)\right\},$$

$$\Sigma_{\delta,3} = \left\{(\lambda, x) \in \Sigma_\delta \;\middle|\; t = \gamma_{\lambda,x}^{-1}R(z, \delta) \in \partial A_{\lambda,x} \text{ and } \gamma_{\lambda,x}'(t) \text{ is tangent to } \partial R(z, \delta)\right\}.$$

Then $\Sigma_{\delta,3} \subset H \times O$ is a smooth submanifold of dimension $2n - 1$ and $\Sigma_{\delta,1}, \Sigma_{\delta,2} \subset H \times O$ are smooth submanifolds of dimension $2n$.

(3) *If $j = 1, 2$ and $p = (\lambda, x) \in \Sigma_{\delta,j}$, then there is an open neighborhood $p \in \mathcal{U} \subset H \times O$ and an open interval $D \subset \mathbb{R}$ containing δ such that $\{\mathcal{U} \cap \Sigma_{\delta',j}\}_{\delta' \in D}$ defines a smooth foliation of $\mathcal{U}$.*

(4) *Let $l : H \times O \to \mathbb{R}$ be the map which sends (λ, x) to the length of $\gamma_{\lambda,x}(A_{\lambda,x}) \cap R(z, \delta)$. Then l is a continuous function.*

Proof. Claims (1) and (3) follow from the theorem of existence and uniqueness of integral curves of smooth vector fields. Statement (2) follows from observing that $\mathcal{V}$ is tangent to $\partial R(z, \delta)$ along a codimension 1 submanifold of $\partial R(z, \delta)$. Finally, (4) follows from the same arguments as in the proof of Lemma 2.3. □

2.2. *J-perturbed gradient segments*

In this section we define multivalued perturbations of the gradient flow equation $\gamma' = -I\mathcal{X}(\gamma)$ in terms of infinitesimal variations of the almost complex structure. These perturbations are called multivalued because they are defined on finite nonramified coverings of the tori

$R(z,\delta)$. Take a point $z \in Z$ and suppose that the isotropy group of z has k elements. For any δ small enough so that $R(z,\delta)$ is a solid torus and smaller than the injectivity radius we define

$$R^\sharp(z,\delta) = \bigl\{(\alpha, x) \in S^1 \times R(z,\delta) \bigm| \alpha^{-1}x \in S(z,\delta)\bigr\}.$$

Then the projection to the second factor

$$\pi : R^\sharp(z,\delta) \to R(z,\delta)$$

is an unramified covering of degree k because the set of elements $\theta \in S^1$ such that $\theta \cdot S(z,\delta) = S(z,\delta)$ coincides with the stabilizer of z (here we use that δ is less than the injectivity radius). On the other hand, if we denote by $\mathcal{O} \subset R(z,\delta)$ the S^1 orbit through z, the covering $\pi^{-1}(\mathcal{O}) \to \mathcal{O}$ is isomorphic to the map $S^1 \to S^1$ which sends θ to θ^k. It follows that $\pi^{-1}(\mathcal{O})$ is connected, and hence so is $R^\sharp(z,\delta)$. Consequently, $R^\sharp(z,\delta)$ is a solid torus. Consider the action of S^1 on $R^\sharp(z,\delta)$ defined as $\theta \cdot (\alpha, x) = (\theta\alpha, \theta \cdot x)$ for any $\theta \in S^1$. This action is free and, with respect to this action, π is equivariant.

Let (V,η) be a symplectic vector space and let $\mathcal{J} \subset \operatorname{End} V$ be the set of complex structures $J \in \operatorname{End} V$ such that $\eta(\cdot, J\cdot)$ defines an Euclidean pairing. The tangent space $T_J\mathcal{J}$ can be identified with the space of endomorphisms $j \in \operatorname{End} V$ satisfying $jJ + Jj = 0$ and $j + j^{*\eta} = 0$, where $j^{*\eta}$ is the dual of j with respect to η. Hence the sections of the vector bundle

$$E = \bigl\{j \in \operatorname{End} TX \bigm| jI + Ij = 0,\ j + j^{*\omega} = 0\bigr\}$$

can be identified with the infinitesimal deformations of I as an almost complex structure compatible with ω. The following lemma is elementary and well known.

Lemma 2.7. *For any $0 \neq v \in V$ and any $J \in \mathcal{J}$ the map $T_J\mathcal{J} \ni j \mapsto jv \in V$ is onto.*

Fix some point $z \in Z$. The previous lemma implies that we can find $j_1, \dots, j_k \in E_z$ such that for any $v \in T_zX$ the vectors $j_1(v), \dots, j_k(v)$ span T_zX. Choose $\delta_z > 0$ smaller than the injectivity radius and the δ in Lemma 2.5, such that $R(z,\delta_z)$ is a solid torus and such that there exist sections $J_1, \dots, J_k \in C^\infty(S(z,\delta_z); E)$ satisfying: (1) $J_i(z) = j_i$ and (2) for any $z' \in S(z,\delta_z)$ and tangent vector $v \in T_{z'}X$ the vectors $J_1(z')v, \dots, J_k(z')v$ span $T_{z'}X$. The pullback vector bundle

$$\pi^*E \to R^\sharp(z,\delta_z)$$

admits a canonical lift of the S^1 action on $R^\sharp$. Since such action is free and $\{1\} \times S(z,\delta) \subset R^\sharp(z,\delta)$ is a slice, one can extend uniquely the sections $J_1, \dots, J_k$ to equivariant sections $J_1^\sharp, \dots, J_k^\sharp$ of the vector bundle π^*E. Denote by

$$\mathbb{J}_z \subset C^\infty\bigl(R^\sharp(z,\delta_z); \pi^*E\bigr)$$

the span of the sections $J_1^\sharp, \dots, J_k^\sharp$. Let $\beta : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ be a smooth nonincreasing function satisfying, for a small $\varepsilon > 0$, $\beta(t) = 1$ if $t < \varepsilon$ and $\beta(t) = 0$ if $t > 1 - \varepsilon$. For any positive $\delta < \delta_z$ denote by $\eta_{z,\delta} : R(z,\delta_z) \to \mathbb{R}_{\geqslant 0}$ the unique invariant function whose restriction to $S(z,\delta_z)$ satisfies $\eta_{z,\delta}(\exp v) = \beta(|v|/\delta)$ for any $v \in T_zX$.

To state the following lemma we need to introduce some notation. Let $\epsilon > 0$ be a small real number, let $\tau > 0$ and let $\gamma : [0, \tau] \to X$ be an ϵ-perturbed gradient segment whose image intersects the interior of $R(z, \delta_z)$. Combining Definition 2.1, the deduction before Lemma 2.2, and Lemma 2.5, we deduce that there exists a connected and simply connected neighborhood $U \subset R(z, \delta_z)$ of $\gamma(A) \cap R(z, \delta_z)$ and a vector field $\mathcal{V}$ on U which is tangent to γ. Choose a lift $\sigma : U \to R^\sharp(z, \delta_z)$. If $j \in \mathbb{J}_z$ is sufficiently near 0, then one can define $\gamma_j : [0, \tau] \to X$ by the properties $\gamma_j(0) = \gamma(0)$ and $\gamma_j'(t) = \mathcal{V}_{\gamma_j(t)} - \eta_{z,\delta}(j \circ \sigma \circ \gamma_j)(t)\mathcal{X}_{\gamma_j(t)}$ (one only needs that, for any $t \in [0, \tau]$, $\gamma_j(t)$ stays in U). Hence $e(j) = \gamma_j(\tau)$ is defined for any j contained in a small neighborhood of 0 in $\mathbb{J}_z$.

Lemma 2.8. *If ϵ is small enough then the map e is differentiable and the differential $de(0) : \mathbb{J}_z \to T_{\gamma(\tau)}$ is onto.*

Proof. It follows from standard results on ODEs and the definition of $\mathbb{J}_z$, using for example the same coordinate charts as in the proof of Lemma 2.3. □

The union of the interiors of the sets $R(z, \delta_z)$ as z runs over all points in Z contains Z, so by compactness one can take points $z_1, \dots, z_s \in Z$ such that Z is contained in the union of the interiors of the sets $R(z_i, \delta_{z_i})$. Let $R_i = R(z_i, \delta_{z_i})$, $\eta_i = \eta_{z_i,\delta_{z_i}}$ and $R_i^\sharp = R^\sharp(z_i, \delta_{z_i})$ and $\mathbb{J}_i = \mathbb{J}_{z_i}$. Assume that ϵ is small enough so that Lemmata 2.5 and 2.8 hold true. For any $j \in \mathbb{J}_i$ define $\|j\| = \sup_{R_i^\sharp} |j| + \sup_{R_i^\sharp} |\nabla(\eta_i j)|$ and let also

$$\mathbb{J} = \{(j_1, \dots, j_s) \in \mathbb{J}_1 \times \cdots \times \mathbb{J}_r \mid \|j_i\| < \epsilon/s \text{ for each } i\}.$$

Let $J = (j_1, \dots, j_s) \in \mathbb{J}$. A *$J$-perturbed gradient segment* is a tuple $(\gamma, \gamma_1^\sharp, \dots, \gamma_s^\sharp)$, where $\gamma : A \to X$ is an ϵ-perturbed gradient segment and each $\gamma_i^\sharp : \gamma^{-1}R_i \to R_i^\sharp$ is a lift of the restriction of γ, so that $\pi \circ \gamma_i^\sharp = \gamma$ holds on $\gamma^{-1}R_i$, satisfying the equation

$$\gamma' = -\Big(I + \sum_i \eta_i j_i(\gamma_i^\sharp)\Big)\mathcal{X}_\gamma. \tag{2.4}$$

If $\gamma^{-1}R_i$ is empty then $\gamma_i^\sharp$ and its contribution in the differential equation can be ignored. Since $J \in \mathbb{J}$ and the functions η_i are everywhere $\leqslant 1$, Eq. (2.4) implies that γ is an ϵ-perturbed gradient segment. Let S^1 act on J-perturbed gradient segments as

$$\theta \cdot (\gamma, \gamma_1^\sharp, \dots, \gamma_s^\sharp) = (\theta \cdot \gamma, \theta \cdot \gamma_1^\sharp, \dots, \theta \cdot \gamma_s^\sharp).$$

Note that $\theta \cdot \gamma_i^\sharp$ is a lift of $\theta \cdot \gamma$ because the covering maps $\pi : R_i^\sharp \to R_i$ are S^1 equivariant, and Eq. (2.4) is preserved because I, j_i and η_i are S^1 invariant.

2.3. We define an *oriented chain of J-perturbed gradient segments* to be any tuple $C = (K, K_1, \dots, K_s, b)$, where $K \subset X$ is a compact subset, each $K_i \subset R_i^\sharp$ is a compact (possibly empty) subset and $b \in K$, subject to the following conditions.

(1) There is a continuous injective map $\rho : B \to X$, where $B \subset \mathbb{R}$ is a compact interval, which induces a homeomorphism between B and K.

(2) For each i, $B_i = \rho^{-1} R_i$ is connected and $\pi : R_i^\sharp \to R_i$ induces a homeomorphism between K_i and $\rho(B_i)$ (the latter set is independent of the parametrization ρ).
(3) The set $\rho^{-1}(F)$ is finite and for each connected component $B' \subset B \setminus \rho^{-1}(F)$ there is a J-perturbed gradient segment $(\gamma : A \to X, \gamma_1^\sharp, \dots, \gamma_s^\sharp)$ and an increasing homeomorphism $g : B' \to A$ such that $\gamma \circ g = \rho|_{B'}$ and the image of $\gamma_i^\sharp$ coincides with K_i.
(4) The point b is either $\rho(\inf B)$ or $\rho(\sup B)$ (this indicates the orientation of the chain of gradient segments).

We call b the *beginning* of C. If $b = \rho(\sup B)$ then we define the *end* of C to be $e = \rho(\inf B)$. Otherwise we define $e = \rho(\sup B)$. Given a chain $C = (K, K_1, \dots, K_s, b)$ we define $d(K_i) = \sup\{d(x, \partial R_i^\sharp) \mid x \in K_i\}$, where $d(x, \partial R_i^\sharp)$ is defined using the pullback to $R_i^\sharp$ of the Riemannian metric on X (if K_i is empty then we set $d(K_i) = 0$). Define the distance between two chains $C = (K, K_1, \dots, K_s, b)$ and $C' = (K', K_1', \dots, K_s', b')$ as

$$d(C, C') = d_H(K, K') + d(b, b') + \sum d_H(K_i, K_i') d(K_i) d(K_i'), \tag{2.5}$$

where d_H denotes the Hausdorff distance between sets, in the first summand using the Riemannian metric on X and in the third summand using its pullback to $R_i^\sharp$ via π.

2.4. *The space of oriented chains of perturbed gradient segments*

Denote by $\mathcal{C}_J$ the set of oriented chains of J-perturbed gradient segments modulo the relation which identifies two chains C, C' whenever $d(C, C') = 0$. Note that if the chains $C = (K, K_1, \dots, K_s, b)$ and $C' = (K', K_1', \dots, K_s', b')$ are different but $d(C, C') = 0$, then $K = K'$ and for any i such that $K_i \neq K_i'$ both K_i and K_i' are contained in the boundary $\partial R_i^\sharp$ and hence, by Lemma 2.5, consist of a unique point each. Take on $\mathcal{C}_J$ the topology induced by the distance d and define an action of S^1 on $\mathcal{C}_J$ componentwise: $\theta \cdot (K, K_1, \dots, K_s, b) = (\theta \cdot K, \theta \cdot K_1, \dots, \theta \cdot K_s, \theta \cdot b)$. One checks that this action maps elements of $\mathcal{C}_J$ to elements of $\mathcal{C}_J$ using the action of S^1 of J-perturbed gradient segments defined above. The proof of the following lemma is straightforward.

Lemma 2.9. *$\mathcal{C}_J$ is compact, the action of S^1 on $\mathcal{C}_J$ is continuous, and the map*

$$(b, e) : \mathcal{C}_J \to X \times X$$

given by sending each $C \in \mathcal{C}_J$ to its beginning and end is continuous.

3. Definition of the class $[\Delta_{\mathbb{C}^*}]$

3.1. For any $J \in \mathbb{J}_\epsilon$ define $\mathcal{C}_J^0 \subset \mathcal{C}_J$ as the set of chains of perturbed gradient segments $(K, K_1, \dots, K_s, b)$ such that $K \cap F = \emptyset$. We say that $C = (K, K_1, \dots, K_s, b)$ is tangent to R_i if $\emptyset \neq K \cap R_i \subset \partial R_i$ which implies by Lemma 2.5 that $K \cap R_i$ is one point. Define $\mathcal{C}_J^{0,0}$ as the set of chains $C \in \mathcal{C}_J^0$ which are not tangent to any R_i. Let o_i be the degree of the covering $\pi : R_i^\sharp \to R_i$ (equivalently, the order of the isotropy group of z_i). We define

$$\text{weight} : \mathcal{C}_J^{0,0} \to \mathbb{Q}$$

by sending $C = (K, K_1, \dots, K_s, b) \in \mathcal{C}_J^{0,0}$ to the product $o_{i_1}^{-1} \dots o_{i_\nu}^{-1}$, where $\{i_1, \dots, i_\nu\}$ is the set of i such that $K \cap R_i \neq \emptyset$ (since $C \in \mathcal{C}_J^{0,0}$ this implies that $K \cap R_i$ contains points in the interior of R_i). The next theorem will be proved in Section 4.1.

Theorem 3.1. *For any $C \in \mathcal{C}_J^0$ there exist oriented connected manifolds $U_1, \dots, U_N$ of real dimension $2n+1$ and continuous maps $\phi_j : U_j \to \mathcal{C}_J^0$ satisfying these properties*:

(1) *For any j the map $\eta_j : U_j \to \mathbb{R} \times X$ which sends $u \in U_j$ to $(h \circ e \circ \phi_j(u), b \circ \phi_j(u))$ is a local diffeomorphism preserving the orientation.*
(2) *The union $\phi_1(U_1) \cup \dots \cup \phi_N(U_N)$ is a neighborhood of C in $\mathcal{C}_J^0$.*
(3) *If $C \in \mathcal{C}_J^{0,0}$ then N can be taken to be* 1.

3.2. We briefly recall the notion of pseudocycle introduced in §6.5 of [11]. Let N be a smooth manifold. A subset $R \subset N$ is said to have dimension at most d if there is a d-dimensional manifold S and a smooth map $g : S \to N$ such that $R \subset g(S)$. Given a smooth map $f : M \to N$ of oriented manifolds, the omega limit set of f, denoted $\Omega_f \subset N$, is the intersection of all closed subsets $\overline{f(M \setminus K)} \subset N$ as K runs over the collection of all compact subsets of M. If M has dimension d, the map $f : M \to N$ is called a d-dimensional pseudocycle if Ω_f has dimension at most $d-2$. Two d-dimensional pseudocycles $f : M \to N$ and $f' : M' \to N$ are called bordant if there is an oriented manifold W of dimension $d+1$ with boundary $\partial W = M \cup (-M')$ and a smooth map $F : W \to N$ extending f and f' such that Ω_F has dimension at most $d-1$. In Remark 6.5.3 of [11] a construction is given which assigns to any d-dimensional homology class $\beta \in H_d(N)$ a bordism class of d-dimensional pseudocycles $f : M \to N$. We say that the $f : M \to N$ represents β.

Recall that two smooth maps $\alpha : M \to N$ and $\alpha' : M' \to N$ are said to be transverse if the map $(\alpha, \alpha') : M \times M' \to N \times N$ is transverse to the diagonal $\Delta_N \subset N \times N$. In this situation, the set $\mathrm{CS}(\alpha, \alpha') := \{(x, x') \in M \times M' \mid \alpha(x) = \alpha'(x')\}$ is a submanifold of $M \times M'$ of dimension $\dim M + \dim M' - \dim N$ (here CS stands for Cartesian Square). If $\alpha : A \to B$ is a submersion, then α is transverse to any smooth map $\alpha' : A' \to B$. The proof of the following lemma is straightforward.

Lemma 3.2. *Suppose that $\alpha : M \to N$ and $\alpha' : M' \to N$ are two transverse maps satisfying $\Omega_\alpha \cap \alpha'(M') = \alpha(M) \cap \Omega_{\alpha'} = \emptyset$. Then $\mathrm{CS}(\alpha, \alpha')$ is a compact submanifold of $M \times M'$.*

3.3. Define the set $\mathcal{P}_J = S^1 \times \mathcal{C}_J$ and let

$$\Theta_J : \mathcal{P}_J \to X \times X \tag{3.1}$$

be the map $\Theta_J(\theta, C) = (\theta \cdot b(C), e(C))$. Considering the action of $S^1 \times S^1$ on $\mathcal{P}_J$ given by $(\alpha, \beta) \cdot (\theta, C) = (\alpha\beta^{-1}\theta, \beta \cdot C)$ and the product action on $X \times X$, the map Θ_J is $S^1 \times S^1$ equivariant. For any natural number Λ let S_Λ be the unit sphere in $\mathbb{C}^{\Lambda+1}$ centered at the origin. Scalar multiplication gives a free action of S^1 on S_Λ and hence a structure of principal circle bundle on the quotient map $S_\Lambda \to \mathbb{CP}^\Lambda = S_\Lambda / S^1$. The bundles $S_\Lambda \to \mathbb{CP}^\Lambda$ provide finite-dimensional approximations of the universal circle fibration. Define $E_\Lambda = S_\Lambda \times S_\Lambda$ and $B_\Lambda = \mathbb{CP}^\Lambda \times \mathbb{CP}^\Lambda$.

The natural projection $p : E_\Lambda \to B_\Lambda$ endows E_Λ with a structure of principal $S^1 \times S^1$ bundle. Since Θ_J is equivariant, it induces a map

$$\Theta_{J,\Lambda} : \mathcal{P}_{J,\Lambda} = E_\Lambda \times_{S^1\times S^1} \mathcal{P}_J \to X^2_\Lambda := E_\Lambda \times_{S^1\times S^1} (X \times X). \tag{3.2}$$

The manifolds B_Λ and $X \times X$ have natural orientations, which induce an orientation on X^2_Λ. We define a cohomology class $[\Delta_{\mathbb{C}^*}]_\Lambda \in H^{2n-2}(X^2_\Lambda)$ in terms of its pairing with homology classes. The following theorem will be proved in Section 4.2.

Theorem 3.3. *Let $\beta \in H_{2n-2}(X^2_\Lambda)$, and let $f : M \to X^2_\Lambda$ be a pseudocycle representing β. Let $D \subset C^\infty(X^2_\Lambda, TX^2_\Lambda)$ be a linear subspace such that for any $p \in X^2_\Lambda$ the evaluation map $D \to T_pX^2_\Lambda$ is onto, and let $\mathbb{D} = \{\exp\gamma \mid \gamma \in D\} \subset \mathrm{Diff}(X^2_\Lambda)$. There exists a residual subset $\mathcal{R} \subset \mathbb{J} \times \mathbb{D}$ such that for any $(J, \xi) \in \mathcal{R}$ we have*:

1. *The set $\mathcal{T}_{J,\xi} = \{(x, y) \in \mathcal{P}_{J,\Lambda} \times M \mid \Theta_{J,\Lambda}(x) = \xi \circ f(y)\}$ is finite.*
2. *For any $(x, y) \in \mathcal{T}_{J,\xi}$ we have $x \in E_\Lambda \times_{S^1\times S^1} (S^1 \times \mathcal{C}^{0,0}_J)$.*
3. *Let $(x, y) \in \mathcal{T}_{J,\xi}$ and let $b = p(x)$. Let $O \subset B_\Lambda$ be a small neighborhood of b. Take a trivialization of $E_\Lambda|_O$ and denote by $\psi : O \times S^1 \times \mathcal{C}_J \to \mathcal{P}_{J,\Lambda}|_O$ the induced homeomorphism. Suppose that $x = \psi(b, \theta, C)$. Let $\phi : U \to \mathcal{C}_J$ be a continuous map as given by Theorem* 3.1*, where U is an oriented $2n + 1$-dimensional manifold and $\phi(U)$ is a neighborhood of C. Endow $V = O \times S^1 \times U$ with its product orientation, and let $\phi_O : V \to O \times S^1 \times \mathcal{C}_J$ be the map* $(\mathrm{Id}_O, \mathrm{Id}_{S^1}, \phi)$. *The differential δ at $((b, \theta, C), y)$ of the map*
$$(\Theta_J \circ \psi \circ \phi_O, \xi \circ f) : V \times M \to X^2_\Lambda$$
is an isomorphism of vector spaces. Define $\sigma(x, y) = 1$ if δ preserves the orientations and $\sigma(x, y) = -1$ otherwise. Define also $\mathrm{weight}(x) = \mathrm{weight}(C)$.
4. *The following number only depends on β and Λ, and not on D, f, J, ξ:*
$$\Delta_\Lambda(\beta) = \sum_{(x,y)\in\mathcal{T}_{J,\xi}} \sigma(x, y)\,\mathrm{weight}(x) \in \mathbb{Q}.$$

3.4. Definition of the biinvariant diagonal class

The map $\Delta_\Lambda : H_{2n-2}(X^2_\Lambda) \to \mathbb{Q}$ defined by the previous theorem is clearly linear and hence is induced by a cohomology class $[\Delta_{\mathbb{C}^*}]_\Lambda \in H^{2n-2}(X^2_\Lambda)$. To compare this class for different values of Λ, note that there is a natural homotopy class of inclusion $B_\Lambda \subset B_{\Lambda+1}$ whose image is the product of two hyperplanes. This inclusion induces $\iota_\Lambda : X^2_\Lambda \to X^2_{\Lambda+1}$. The same ideas as in the proof of (4) of Theorem 3.3 imply that

$$\iota^{2n-2}_\Lambda [\Delta_{\mathbb{C}^*}]_{\Lambda+1} = [\Delta_{\mathbb{C}^*}]_\Lambda.$$

For big enough Λ the map ι^{2n-2}_Λ is an isomorphism and we can identify the cohomology groups $H^{2n-2}(X^2_\Lambda) \simeq H^{2n-2}_{S^1\times S^1}(X \times X)$. Hence the class $[\Delta_{\mathbb{C}^*}]_\Lambda$ defines an equivariant cohomology class

$$[\Delta_{\mathbb{C}^*}] \in H^{2n-2}_{S^1\times S^1}(X \times X).$$

We call $[\Delta_{\mathbb{C}^*}]$ the biinvariant diagonal class.

4. Parametrizing oriented J-perturbed chains of gradient segments

4.1. Proof of Theorem 3.1

Let $C = (K, K_1, \dots, K_s, b) \in \mathcal{C}_J^0$ and assume that the J-perturbed gradient segment $(\gamma : A \to X, \gamma_1^\sharp, \dots, \gamma_s^\sharp)$ parameterizes C, so $A \subset \mathbb{R}$ is a compact interval, $K = \gamma(A)$ and $K_i = \gamma_i^\sharp(\gamma^{-1} R_i)$. Assume that $b = \gamma(\sup A)$. Let $\mathcal{I}'$ be the set of i's such that K intersects the interior of R_i and let $\mathcal{I}''$ be the set of i's such that C is tangent to R_i. Choose for any $i \in \mathcal{I}'$ a small open neighborhood $M_i^\sharp \subset R_i^\sharp$ of K_i such that $\pi : M_i^\sharp \to M_i := \pi(M_i^\sharp)$ is a diffeomorphism of open manifolds with boundary, and let $\sigma_i : M_i \to M_i^\sharp$ be its inverse. For any $i \in \mathcal{I}''$ let $q_i = K \cap R_i$, which by Lemma 2.5 consists of a unique point, and let $Q_i = \pi^{-1}(q_i) \subset R_i^\sharp$, which consists of o_i different points. Let $\mathcal{Q} = \prod_{i \in \mathcal{I}''} Q_i$. Given $q = (q_i) \in \mathcal{Q}$, choose for each $i \in \mathcal{I}''$ a small open neighborhood $M_i^\sharp \subset R_i^\sharp$ of q_i such that $\pi : M_i^\sharp \to M_i := \pi(M_i^\sharp)$ is a diffeomorphism of open manifolds with boundary, and let $\sigma_i : M_i \to M_i^\sharp$ be its inverse. Let $\mathcal{I} = \mathcal{I}' \cup \mathcal{I}''$. For any $q \in \mathcal{Q}$ let $M_q \subset X$ be an open neighborhood of $\gamma(A)$ such that $M_q \cap R_i \subset M_i$ for any $i \in \mathcal{I}$. Define the following vector field on M_q

$$\mathcal{V}_q = -\left(I + \sum_{i \in \mathcal{I}} \eta_i j_i(\sigma_i)\right)\mathcal{X}. \tag{4.1}$$

Then $\gamma : A \to X$ is an integral curve of $\mathcal{V}_q$. Furthermore, all the integral curves of $\mathcal{V}_q$ satisfy the conditions of Definition 2.1. Let $\lambda_0 = h(\gamma(\inf A))$ and $x_0 = \gamma(\sup A)$. Applying Lemma 2.6 to $\mathcal{V}_q$ we obtain an open neighborhood $U_q = H_q \times O_q \subset \mathbb{R} \times X$ of (λ_0, x_0) and for each $(\lambda, x) \in U_q$ an integral curve $\gamma_{\lambda,x} : A_{\lambda,x} \to X$, which is an ϵ-perturbed gradient segment. Let $K_{\lambda,x} = \gamma_{\lambda,x}(A_{\lambda,x})$ and let $K_{\lambda,x,i} = K_{\lambda,x} \cap R_i$. Taking U_q small enough we can assume that $K_{\lambda,x,i}$ is nonempty if and only if $i \in \mathcal{I}$. Define

$$\phi_q(\lambda, x) = \big(K_{\lambda,x}, K_{\lambda,x,1}, \dots, K_{\lambda,x,s}, \gamma_{\lambda,x}(\sup A_{\lambda,x})\big). \tag{4.2}$$

Then $\phi_q(\lambda, x) \in \mathcal{C}_J^0$, because it can be parametrized by the J-perturbed gradient segment $(\gamma_{\lambda,x}, \gamma_{\lambda,x,1}, \dots, \gamma_{\lambda,x,s})$, where $\gamma_i^\sharp : \gamma_{\lambda,x}^{-1} R_i \to R_i^\sharp$ is equal to $\sigma_i \circ \gamma_i$. In this way we have defined a continuous map $\phi_q : U_q \to \mathcal{C}_J^0$. Picking the right orientation of U_q claim (1) of Theorem 3.1 holds trivially. We prove that $\bigcup_{q \in \mathcal{Q}} \phi_q(U_q)$ is a neighborhood of C in $\mathcal{C}_J^0$, which is claim (2) of the theorem. By (1) in Lemma 2.5 for any $C' = (K', K_1', \dots, K_s', b') \in \mathcal{C}_J^0$ each intersection $K' \cap R_i$ is connected, so if C' lies sufficiently near C the compact K' must be an integral curve of one of the vector fields $\mathcal{V}_q$. When $C \in \mathcal{C}_J^{0,0}$ the set $\mathcal{I}''$ is empty, so there is a unique open set U and map $\phi : U \to \mathcal{C}_J^{0,0}$ whose image is a neighborhood of C. This proves (3). Finally, to deal with the case $b = \gamma(\inf A)$ we proceed exactly as before, replacing the last entry in (4.2) by x.

4.2. Proof of Theorem 3.3

Let $\mathcal{C} = \{(J, C) \mid J \in \mathbb{J},\ C \in \mathcal{C}_J\}$. Define the distance between points in $\mathcal{C}$ as $d((J, C), (J', C')) = \|J - J'\| + d(C, C')$, where if $J = (j_1, \dots, j_s)$ and $J' = (j_1', \dots, j_s')$ then $\|J - J'\| =$

$\|j_1 - j_1'\| + \cdots + \|j_s - j_s'\|$ and $d(C, C')$ is defined as in (2.5). Consider on $\mathcal{C}$ the topology induced by this distance and define the maps

$$(b, e) : \mathcal{C} \to X \times X$$

by mapping $(J, C) \in \mathcal{C}$ to $(b(C), e(C))$, where $b(C), e(C)$ are defined in Section 2.3. Consider also the projection

$$\pi_J : \mathcal{C} \to \mathbb{J}$$

sending any $(J, C) \in \mathcal{C}$ to J. For any integer r let $\mathcal{C}^r = \{(J, (K, \ldots)) \in \mathcal{C} \mid \sharp K \cap F = r\}$ be the set of perturbed chains which meet the fixed point set at r points. Fix from now on an orientation of $\mathbb{J}$.

Lemma 4.1. *Let $\mathbf{K} = (J, (K, \ldots)) \in \mathcal{C}^0$ and let $\mathcal{L} = \{l \mid K \cap \operatorname{int} R_l = \emptyset,\ K \cap \partial R_l \neq \emptyset\}$. For any $l \in \mathcal{L}$ let $q_l \in R_l$ be the unique point of intersection of K with R_l (see (2) in Lemma 2.5), and let $Q_l \subset R_l^\sharp$ be the preimage of q_l. Let $\mathcal{Q} = \prod_{l \in \mathcal{L}} Q_l$.*

(1) *There exists a collection of connected oriented open manifolds $\{\mathcal{U}_q\}_{q \in \mathcal{Q}}$ of dimension equal to $2n + 1 + \dim \mathbb{J}$ and continuous maps $\Phi_q : \mathcal{U}_q \to \mathcal{C}$ such that the union $\bigcup_{q \in \mathcal{Q}} \Phi_q(\mathcal{U}_q)$ is a neighborhood of $\mathbf{K}$ in $\mathcal{C}$.*
(2) *For any q both $\pi_J \circ \Phi_q : \mathcal{U}_q \to \mathbb{J}$ and $(b, e) \circ \Phi_q : \mathcal{U}_q \to X \times X$ are smooth maps.*
(3) *For any $l \in \mathcal{L}$ define $\mathcal{O}_l := \{(J, (K, \ldots)) \in \mathcal{C} \mid K \cap \operatorname{int} R_l = \emptyset\}$. Define also, for any q, $\mathcal{O}_{q,l} = \Phi_q^{-1}(\mathcal{O}_l)$. If $q = (q_l) \neq q' = (q_l')$, then we have $\Phi_q^{-1}(\Phi_{q'}(\mathcal{U}_{q'})) = \bigcap_{q_l \neq q_l'} \mathcal{O}_{q,l}$. The boundary $\partial \mathcal{O}_{q,l} \subset \mathcal{U}_q$ is the disjoint union of smooth submanifolds $\mathcal{S}_{q,l,1}, \mathcal{S}_{q,l,2}, \mathcal{S}_{q,l,3}$ of codimensions 1, 1 and 2 respectively.*

Proof. (1) and (2) follow from the same arguments as the proof of Theorem 3.1 given in Section 4.1, replacing X by $\mathbb{J} \times X$ and choosing the perturbations j_i in (4.1) using the coordinate in $\mathbb{J}$ (note that $\mathcal{L}$ corresponds to $\mathcal{I}''$). The first statement in (3) follows from the construction; in the second statement, the submanifolds $\mathcal{S}_{q,j,i}$ are the analogues of the submanifolds $\Sigma_{\delta,i}$ in Lemma 2.6. □

Lemma 4.2. *Let $r \geqslant 1$ be an integer. There is a countable collection of connected smooth manifolds $\{\mathcal{V}_{r,i}\}_{i \in \mathbb{N}}$ of dimension $2n + 1 - r$ and continuous maps $\Psi_{r,i} : \mathcal{U}_{r,i} \to \mathcal{C}^r$ such that*: (1) *the union $\bigcup_{i \in \mathbb{N}} \Psi_{r,i}(\mathcal{V}_{r,i})$ is equal to $\mathcal{C}^r$*, (2) *for each i the compositions $(b, e) \circ \Psi_{r,i} : \mathcal{V}_{r,i} \to X \times X$ and $\pi_J \circ \Psi_{r,i} : \mathcal{V}_{r,i} \to \mathbb{J}$ are smooth maps.*

Proof. Given a closed interval $[u, v] \subset \mathbb{R}$ we define $\mathcal{C}([u, v]) \subset \mathcal{C}$ as the subset of all $\mathbf{K} \in \mathcal{C}$ such that $h(b(\mathbf{K})) = u$ and $h(e(\mathbf{K})) = v$. More generally, for any interval $A \subset \mathbb{R}$, let $\mathcal{C}(A)$ to be the union of all the sets $\mathcal{C}(B)$ as B runs over the collection of the compact subintervals of A. Define also for any r the set $\mathcal{C}^r(A) = \mathcal{C}^r \cap \mathcal{C}(A)$. We prove the lemma in several steps.

Step 1. Let $\mathcal{C}^{\geqslant r} = \bigsqcup_{r' \geqslant r} \mathcal{C}^{r'}$. It is straightforward to check (as in Lemma 2.9) that the projection $\pi_J : \mathcal{C}^{\geqslant r} \to \mathbb{J}$ is proper. Furthermore, $\mathcal{C}^r$ is open in $\mathcal{C}^{\geqslant r}$ so, defining for any integer α the subset $\mathcal{C}^{r,\alpha} = \{\mathbf{K} \in \mathcal{C}^r \mid d(\mathbf{K}, \mathcal{C}^{\geqslant r+1}) \in [2^{-\alpha}, 2^{-\alpha+1}]\} \subset \mathcal{C}^r$, the restriction of π_J to each $\mathcal{C}^{r,\alpha}$ is proper, and also $\mathcal{C}^r = \bigcup_\alpha \mathcal{C}^{r,\alpha}$. Hence it suffices to construct for any $\mathbf{K} \in \mathcal{C}^r$ a collection of connected manifolds $\mathcal{V}_1, \ldots, \mathcal{V}_p$ of dimension $2n+1-r$ and continuous maps $\Psi_i : \mathcal{V}_i \to \mathcal{C}^r$ satisfying (2) of the lemma and such that $\Psi_1(\mathcal{V}_1) \cup \cdots \cup \Psi_p(\mathcal{V}_p)$ is a neighborhood of $\mathbf{K}$ in $\mathcal{C}^r$.

Step 2. Let $A = [u, v] \subset \mathbb{R}$ be a compact interval such that $A \cap \operatorname{int} h(Z') \neq \emptyset$ and let $\mathbf{K} \in \mathcal{C}^0(A)$. We claim that there exist connected open manifolds $\mathcal{V}_1, \ldots, \mathcal{V}_r$ of dimension equal to $2n - 1 + \dim \mathbb{J}$ and continuous maps $\Psi_i : \mathcal{V}_i \to \mathcal{C}(A)$ satisfying: (1) the union $\Psi_1(\mathcal{V}_1) \cup \cdots \cup \Psi_r(\mathcal{V}_r)$ is a neighborhood of $\mathbf{K}$ in $\mathcal{C}(A)$, (2) for any i the composition $\pi_J \circ \Psi_i : \mathcal{V}_i \to \mathbb{J}$ is a smooth map, and (3) the map $(b \circ \Psi_i, e \circ \Psi_i) : \mathcal{V}_i \to h^{-1}(u) \times h^{-1}(v)$ is a smooth submersion for each i. Except from (3) everything follows as in the proof of Lemma 4.1, and (3) is a consequence of Lemma 2.8. Similar statements hold replacing $[u, v]$ by $(u, v]$ and $[u, v)$, replacing the map $(b \circ \Psi_i, e \circ \Psi_i)$ by $b \circ \Psi_i$ in the first case and by $e \circ \Psi_i$ in the second one, and decreasing the dimensions of $\mathcal{V}_i$ one unit.

Step 3. Let again $A = [u, v] \subset \mathbb{R}$ be a compact interval, and let $\mathbf{K} = (J, (K, \ldots)) \in \mathcal{C}^1(A)$. We claim that there exist connected open manifolds $\mathcal{V}_1, \ldots, \mathcal{V}_r$ of dimension equal to $2n - 2 + \dim \mathbb{J}$ and continuous maps $\Psi_i : \mathcal{V}_i \to \mathcal{C}(A)$ satisfying: (1) the union $\Psi_1(\mathcal{V}_1) \cup \cdots \cup \Psi_r(\mathcal{V}_r)$ is a neighborhood of $\mathbf{K}$ in $\mathcal{C}(A)$, (2) for any i the compositions $\pi_J \circ \Psi_i : \mathcal{V}_i \to \mathbb{J}$ and $(b \circ \Psi_i, e \circ \Psi_i) : \mathcal{V}_i \to h^{-1}(u) \times h^{-1}(v)$ are smooth maps. This can be proved as in the proof of Lemma 4.1 using the (un)stable manifold theorem for critical sets on normally hyperbolic vector fields (see e.g. Theorem 4.1 in [6]). To be more concrete, suppose for simplicity that $A \cap h(Z') = \emptyset$, let $\mathbf{K} = (J, (K, \ldots)) \in \mathcal{C}^1(A)$ and let $F' \subset F$ be the connected component to which $K \cap F$ belongs. Let $\pi_\pm : W_\pm \to F'$ be the (un)stable manifolds and the corresponding submersions for the vector field $-I\mathcal{X}$. Then one can identify a neighborhood of $\mathbf{K}$ in $\mathcal{C}^1(A)$ with $(W_+ \times_{F'} W_-) \cap h^{-1}(u) \times h^{-1}(v)$.

Step 4. Now let $r \geqslant 1$ be any integer and let $\mathbf{K} = (J, (K, \ldots)) \in \mathcal{C}^r$. Assume that $h(b(\mathbf{K})) < h(e(\mathbf{K}))$, the other cases (either the opposite inequality or equality) being analogous. Let $K \cap F = \{f_1, \ldots, f_r\}$, labelled in such a way that $d_1 = h(f_1) < \cdots < d_r = h(f_r)$. Let $\eta > 0$ be small enough so that each $A_j := [d_j - \eta, d_j + \eta]$ is disjoint from $h(Z')$. Assume also for simplicity that $h(b(\mathbf{K})) < d_1 - \eta$ and $h(e(\mathbf{K})) > d_r + \eta$. Define the intervals $A'_0 = (h(b(\mathbf{K})) - \eta, d_1 - \eta]$, $A'_r = [d_r + \eta, h(e(\mathbf{K})) + \eta)$ and, for any $1 \leqslant j \leqslant r - 1$, $A'_j = [d_{j-1} + \eta, d_j - \eta]$. Let $X_{j,\pm} = h^{-1}(d_j \pm \eta)$. Then the following fiber product gives a neighborhood of $\mathbf{K}$ in $\mathcal{C}^r$:

$$\mathcal{C}^0(A'_0) \times_{X_{1,-}} \mathcal{C}^1(A_1) \times_{X_{1,+}} \mathcal{C}^0(A'_1) \times_{X_{2,-}} \mathcal{C}^1(A_2) \times \cdots \times \mathcal{C}^1(A_r) \times_{X_{r,+}} \mathcal{C}^0(A'_r).$$

Here the fiber product is defined using the maps $(b, e) : \mathcal{C}^1(A_j) \to X_{j,-} \times X_{j,+}$ and $(b, e) : \mathcal{C}^0(A_j) \to X_{j,+} \times X_{j+1,-}$ for $1 \leqslant j \leqslant r - 1$, the cases $j = 0, r$ being the obvious generalizations. Using the results in Steps 2 and 3 and a simple computation with dimensions, the result follows. $\square$

By the same argument as in Step 1 of the proof of Lemma 4.2, one can choose a countable set $\{\mathbf{K}_\nu\} \subset \mathcal{C}^0$ such that, denoting by $\{\mathcal{U}_{\nu,q}\}_{q \in \mathcal{Q}_\nu}$ the manifolds and by $\Phi_{\nu,q} : \mathcal{U}_{\nu,q} \to \mathcal{C}$ the maps constructed in Lemma 4.1 for $\mathbf{K} = \mathbf{K}_\nu$, the images $\Phi_{\nu,q}(\mathcal{U}_{\nu,q})$ cover $\mathcal{C}^0$. Let also $\mathcal{S}_{\nu,q,l,i} \subset \mathcal{U}_{\nu,q}$ denote the submanifolds given by the lemma.

Let $\mathcal{M} = \mathbb{D} \times M$. Let $g : N \to X^2_\Lambda$ be a smooth map, where N is a smooth manifold of dimension $\leqslant 2n - 4$, satisfying $\Omega_f \subset g(N)$. Define $\mathcal{N} = \mathbb{D} \times N$. Consider the maps

$$F : \mathcal{M} \to X^2_\Lambda, \qquad G : \mathcal{N} \to X^2_\Lambda$$

defined as $F(\xi, m) = \xi \circ f(m)$ and $G(\xi, n) = \xi \circ g(n)$. Define also $\mathcal{P} = S^1 \times \mathcal{C}$ and $\mathcal{P}_\Lambda = E_\Lambda \times_{S^1 \times S^1} \mathcal{P}$, and let the map $\Theta_\Lambda : \mathcal{P}_\Lambda \to X^2_\Lambda$ be defined generalizing in the obvious way the map $\Theta_{J,\Lambda}$ in (3.2). Choose a covering of B_Λ by open sets $\{\mathcal{Z}_\lambda\}$ in such a way that there exist trivializations $E_\Lambda|_{\mathcal{Z}_\lambda} \simeq \mathcal{Z}_\lambda \times S^1$, and denote by $\zeta_\lambda : \mathcal{Z}_\lambda \times S^1 \times \mathcal{C} \to \mathcal{P}_\Lambda|_{\mathcal{Z}_\lambda}$ the induced trivializations of the bundle $\mathcal{P}_\Lambda$. Both maps F and G are submersions, so they are transverse to the following compositions of smooth maps:

$$e_{\lambda,\nu,q} : \mathcal{Z}_\lambda \times S^1 \times \mathcal{U}_{\nu,q} \xrightarrow{\mathrm{Id} \times \mathrm{Id} \times \Phi_{\nu,q}} \mathcal{Z}_\lambda \times S^1 \times \mathcal{C} \xrightarrow{\zeta_\lambda} \mathcal{P}_\Lambda \xrightarrow{\Phi_\Lambda} X^2_\Lambda,$$

and

$$e'_{\lambda,r,i} : \mathcal{Z}_\lambda \times \{1\} \times \mathcal{V}_{r,i} \xrightarrow{\mathrm{Id} \times \iota \times \Psi_{r,i}} \mathcal{Z}_\lambda \times S^1 \times \mathcal{C} \xrightarrow{\zeta_\lambda} \mathcal{P}_\Lambda \xrightarrow{\Phi_\Lambda} X^2_\Lambda,$$

where $\iota : \{1\} \to S^1$ is the inclusion. For the same reason F and G are transverse to the restriction of $e_{l,\nu,q}$ to each of the manifolds $\mathcal{Z}_\lambda \times S^1 \times \mathcal{S}_{\nu,q,l,i}$. Hence we have six countable sequences of smooth manifolds: $\mathrm{CS}(e_{\lambda,\nu,q}, F)$, $\mathrm{CS}(e_{\lambda,\nu,q}|_{\mathcal{Z}_\lambda \times S^1 \times \mathcal{S}_{\nu,q,l,i}}, F)$, $\mathrm{CS}(e'_{\lambda,r,i}, F)$, and the same ones replacing F by G. We denote by **CS** the collection of all these manifolds. Each of the manifolds in **CS** projects smoothly to $\mathbb{J} \times \mathbb{D}$, and by Sard's theorem there exists a residual subset $\Omega \subset \mathbb{J} \times \mathbb{D}$ of regular values of all these maps. Furthermore,

$$\dim \mathrm{CS}(e_{\lambda,\nu,q}, F) = \dim \mathbb{J} + \dim \mathbb{D},$$
$$\dim \mathrm{CS}(e_{\lambda,\nu,q}|_{\mathcal{Z}_\lambda \times S^1 \times \mathcal{S}_{\nu,q,l,i}}, F) = \dim \mathbb{J} + \dim \mathbb{D} - 1 \quad \text{for } i = 1, 2,$$

and all the remaining manifolds in **CS** have dimension $\leqslant \dim \mathbb{J} + \dim \mathbb{D} - 2$.

Lemma 4.3. *Define for any $J \in \mathbb{J}$ the preimage $\mathcal{V}_{J,r,i} := (\pi_J \circ \Psi_{r,i})^{-1}(J)$. The omega-limit set $\Omega_{\Theta_{J,\Lambda}}$ is contained in the union of the sets $e'_{\lambda,r,i}(\mathcal{Z}_i \times \{1\} \times \mathcal{V}_{J,r,i})$.*

Proof. Let $\{\mathbf{x}_i\} \subset \mathcal{P}_{J,\Lambda}$ be a diverging sequence such that $\Theta_{J,\Lambda}(\mathbf{x}_i)$ converges in X^2_Λ. Recall that $p : X^2_\Lambda \to B_\Lambda$ denotes the projection. Passing to a subsequence we may assume that $\{p(\mathbf{x}_i)\} \subset \mathcal{Z}_\lambda$ for some λ, so that we can write $\zeta_\lambda^{-1}(\mathbf{x}_i) = (\beta_i, \theta_i, \mathbf{K}_i) \in \mathcal{Z}_\lambda \times S^1 \times \mathcal{C}_J$. Passing again to a subsequence we may assume that $\beta_i \to \beta$, $\theta_i \to \theta$ and $\mathbf{K}_i \to \mathbf{K}$. Suppose that $\mathbf{K} = (J, (K, K_1, \ldots, K_s, b))$. Since $\{\mathbf{x}_i\}$ diverges, $\mathbf{K}$ does not belong to $\mathcal{C}^0_J$ and consequently $\{y_1, \ldots, y_r\} := K \cap F \neq \emptyset$, where we may suppose that $h(y_1) < \cdots < h(y_r)$. There are two cases to consider, either $h(b) \leqslant h(y_1)$ or $h(y_r) \leqslant h(b)$. In the first case consider the (discontinuous) map $\rho : X \to X$ defined by $\rho(x) = x$ if $h(x) > h(y_1)$ and $\rho(x) = \theta \cdot x$ if $h(x) \leqslant h(y_1)$. This map lifts to maps $\rho : R^\sharp_i \to R^\sharp_i$. Define $\mathbf{K}' = (J, (K', K'_1, \ldots, K'_s, b'))$ by setting $K' = \rho(K)$, $K'_i = \rho(K_i)$, $b' = \rho(b) = \theta \cdot b$. It turns out that $\mathbf{K}' \in \mathcal{C}_J$ and that $\Phi_{J,\Lambda} \circ \zeta_l(\beta, \theta, \mathbf{K}) = \Phi_{J,\Lambda} \circ \zeta_l(\beta, 1, \mathbf{K}')$, so the result follows from (1) in Lemma 4.2. The case $h(y_r) \leqslant h(b)$ is dealt with similarly. $\square$

Combining the previous lemma with the estimates above on the dimensions of the manifolds in **CS**, together with Lemma 3.2, it follows that for $(J,\xi)\in\mathcal{R}$ the space $\mathcal{T}_{J,\xi}$ is a zero dimensional compact manifold. This proves (1) of Theorem 3.3. Claims (2) and (3) also follow from the estimate on the dimension. To prove claim (4), suppose that $(J,\xi),(J',\xi')\in\mathcal{R}$. By standard arguments, there exists a smooth path $\gamma:[0,1]\to\mathbb{J}\times\mathbb{D}$ going from (J,ξ) to (J',ξ') which is transverse to the projections to $\mathbb{J}\times\mathbb{D}$ from each of the manifolds in **CS**. This implies (using again the estimates on dimensions, Lemmas 3.2 and 4.3) that

$$\mathcal{T}=\bigl\{(x,y,t)\in\mathcal{P}_\Lambda\times\mathcal{M}\times[0,1]\bigm|\pi(x,y)=\gamma(t),\ \Theta_\Lambda(x)=F(y)\bigr\}$$

is compact oriented graph (here $\pi:\mathcal{P}_\Lambda\times\mathcal{M}\to\mathbb{J}\times\mathbb{D}$ is the projection). More precisely, there is a decomposition $\mathcal{T}=\mathcal{T}_{\text{edge}}\cup\mathcal{T}_{\text{vertex}}\cup\mathcal{T}_\partial$, where

$$\mathcal{T}_{\text{vertex}}=\bigl\{(x,y,t)\in\mathcal{T}\bigm|x\in\zeta_\lambda\bigl(\mathcal{Z}_\lambda\times S^1\times\mathcal{S}_{\nu,q,l,i}\bigr)\text{ for some }\lambda,q,l\text{ and }i\in\{1,2\}\bigr\},$$

and hence, by transversality, is a finite set, and

$$\mathcal{T}_\partial=\bigl\{(x,y,t)\in\mathcal{T}\bigm|t\in\{0,1\}\bigr\}=\mathcal{T}_{J,\xi}\cup\mathcal{T}_{J',\xi'}.$$

Finally, $\mathcal{T}_{\text{edge}}=\mathcal{T}\setminus(\mathcal{T}_{\text{vertex}}\cup\mathcal{T}_\partial)$ is an oriented 1-manifold, so each of its connected components (which we call edges) γ has a beginning and an end, $\operatorname{begin}(\gamma),\operatorname{end}(\gamma)\in\mathcal{T}_{\text{vertex}}\cup\mathcal{T}_\partial$. For each $p\in\mathcal{T}_{\text{vertex}}\cup\mathcal{T}_\partial$ there is a positive integer k, which is equal to 1 if and only if $p\in\mathcal{T}_\partial$, and a neighborhood of p in $\mathcal{T}$ homeomorphic to a neighborhood of 0 in $\{z\in\mathbb{C}\mid z^k\in\mathbb{R}_{\geqslant 0}\}$. In particular any $p\in\mathcal{T}_\partial$ is an extreme (either beginning or end) of a unique edge. Denote, for any $p=(x,y,t)\in\mathcal{T}_\partial$, $\sigma(p)=\sigma(x,y)$. Reversing the orientation of $\mathcal{T}$ if necessary, we may assume the following. For any $p\in\mathcal{T}_{J,\xi}\subset\mathcal{T}_\partial$ belonging to the extremes of γ, $\sigma(p)=1$ if $p=\operatorname{begin}(\gamma)$, and $\sigma(p)=-1$ if $p=\operatorname{end}(\gamma)$; similarly, for any $p'\in\mathcal{T}_{J',\xi'}\subset\mathcal{T}_\partial$ belonging to the extremes of γ', $\sigma(p')=-1$ if $p'=\operatorname{begin}(\gamma')$, and $\sigma(p')=1$ if $p'=\operatorname{end}(\gamma')$. Denote, for any $p=(x,y,t)\in\mathcal{T}_{\text{edge}}$, $\operatorname{weight}(p)=\operatorname{weight}(x)$. This defines a locally constant function on $\mathcal{T}_{\text{edge}}$ so for any $\gamma\in\pi_0(\mathcal{T}_{\text{edge}})$ we have $\operatorname{weight}(\gamma)\in\mathbb{Q}$. For any $p\in\mathcal{T}_{\text{vertex}}$ we have $\sum_{\operatorname{begin}(\gamma)=p}\operatorname{weight}(\gamma)=\sum_{\operatorname{end}(\gamma)=p}\operatorname{weight}(\gamma)$, where both sums run over the set of edges. Putting together all these observations, we deduce that

$$\sum_{(x,y)\in\mathcal{T}_{J,\xi}}\sigma(x,y)\operatorname{weight}(x)=\sum_{(x,y)\in\mathcal{T}_{J',\xi'}}\sigma(x,y)\operatorname{weight}(x),$$

which is what we wanted to prove. The same ideas allow to prove that $\Delta_\Lambda(\beta)$ is independent of the chosen pseudocycle $f:M\to X^2_\Lambda$. Finally, if two subspaces $D,D'\subset C^\infty(X^2_\Lambda,TX^2_\Lambda)$ satisfy the requirements of the theorem, then so does $D+D'$, and this allows to prove that the definition of $\Delta_\Lambda(\beta)$ is independent of the choice of D.

5. Proofs of Theorem 1.1, Corollary 1.2 and Theorem 1.4

5.1. *Proof of Theorem 1.1*

Without loss of generality we can assume that m is not contained in $h(Z')$. Let Λ be big enough so that for any $\Lambda'\geqslant\Lambda$ the inclusion $\iota_{\Lambda'}:X^2_{\Lambda'}\to X^2_{\Lambda'+1}$ induces an isomorphism between $(2n-2)$-dimensional cohomology groups. Then we can identify $H^{2n-2}_{S^1\times S^1}(X\times X)$ with

$H^{2n-2}(X^2_\Lambda)$ in such a way that $[\Delta_{\mathbb{C}^*}]$ corresponds to $[\Delta_{\mathbb{C}^*}]_\Lambda$. Let $J \in \mathbb{J}$ be arbitrary. Pick some small $\epsilon > 0$ so that $(m-\epsilon, m+\epsilon)$ is disjoint from $h(Z')$. Then the subset

$$\mathcal{P}^{m,\epsilon}_{J,\Lambda} = \bigl\{x \in \mathcal{P}_J \bigm| (h \times h)\bigl(\Phi_{J,\Lambda}(x)\bigr) \in (m-\epsilon, m+\epsilon)^2\bigr\}$$

carries a natural structure of smooth manifold because any point $x \in \mathcal{P}^{m,\epsilon}_{J,\Lambda}$ is contained in $E_\Lambda \times_{S^1 \times S^1} (S^1 \times \mathcal{C}^{0,0}_J)$ and (3) in Theorem 3.1 (combined with local trivializations of $E_\Lambda \to B_\Lambda$) provides local charts of neighborhoods of x. Furthermore, $\mathcal{P}^{m,\epsilon}_{J,\Lambda}$ is an open neighborhood of $\mathcal{P}^m_{J,\Lambda} = H^{-1}(m,m)$, where $H : \mathcal{P}^{m,\epsilon}_{J,\Lambda} \to (m-\epsilon, m+\epsilon)^2$ is the map sending any $x \in \mathcal{P}^{m,\epsilon}_{J,\Lambda}$ to $(h \times h)(\Phi_{J,\Lambda}(x))$. Since H is a submersion, $\mathcal{P}^m_{J,\Lambda}$ is a smooth manifold and the map

$$\Phi_{J,\Lambda} : \mathcal{P}^m_{J,\Lambda} \to h^{-1}(m)^2_\Lambda = E_\Lambda \times_{S^1 \times S^1} \bigl(h^{-1}(m) \times h^{-1}(m)\bigr) \tag{5.1}$$

represents as a pseudocycle the image of the cohomology class $[\Delta_{\mathbb{C}^*}]_\Lambda$ under the restriction map $H^*(X^2_\Lambda) \to H^*(h^{-1}(m)^2_\Lambda)$. The map (5.1) is an immersion, because $h^{-1}(m)$ contains no fixed points (it fails to be injective at the preimages of pairs $(x, y) \in h^{-1}(m)^2_\Lambda$ where x, y belong to an orbit whose stabiliser is nontrivial). Hence the pseudocycle represented by (5.1) can be identified with the Poincaré dual (PD) of the homology class represented by

$$\Delta_{m,\Lambda} = \bigl\{(x, y) \in h^{-1}(m)^2_\Lambda \bigm| S^1 \cdot x = S^1 \cdot y\bigr\}.$$

Consider the map

$$\pi : h^{-1}(m)^2_\Lambda \to B_\Lambda \times Y_m \times Y_m \to Y_m \times Y_m,$$

where the first map is induced by the quotient $h^{-1}(m) \to h^{-1}(m)/S^1 = Y_m$ and the latter map is the projection, and denote by

$$f : H^{2n+2}(Y_m \times Y_m) \to H^{2n+2}\bigl(h^{-1}(m)^2_\Lambda\bigr)$$

the morphism induced by π, which for Λ big enough is an isomorphism. Since π is a submersion of orbifolds and we can identify $\Delta_{m,\Lambda} = \pi^{-1}(\Delta_m)$, we have

$$PD\bigl([\Delta_{m,\Lambda}]\bigr) = f PD\bigl([\Delta_m]\bigr).$$

That this standard argument in differential topology works in the context of orbifolds follows from the realization of Poincaré duality in terms of differentiable forms, as in §6 of [1] (recall that the de Rham complex for orbifolds is defined in terms of smooth invariant forms on local uniformizers which patch together in the obvious sense). This finishes the proof of Theorem 1.1.

5.2. *Proof of Corollary 1.2*

We will need the following result.

Lemma 5.1. *Take any decomposition* $[\Delta_m] = \sum e_i \otimes f^i \in H^*(Y_m) \otimes H^*(Y_m)$. *For any cohomology class* $a \in H^*(Y_m)$ *we have*

$$\sum \Bigl(\int_{Y_m} \alpha \cup e_i \Bigr) f^i = a. \tag{5.2}$$

Proof. This is well known in the case of smooth manifolds. The same proof as given, for example, on p. 127 of [1] translates word by word to the context of orbifolds via the use of local uniformizers. □

We now prove Corollary 1.2. Take any decomposition

$$[\Delta_{\mathbb{C}^*}] = \sum \epsilon_i \otimes \eta^i \in H^*_{S^1}(X) \otimes H^*_{S^1}(X)$$

and let $a \in H^*(Y_m)$ be any cohomology class. The Kirwan map is compatible with Künneth in the following sense: given any class $\delta \in H^*_{S^1 \times S^1}(X \times X)$, if we write $\delta = \sum \alpha_i \otimes \beta^i$ using the decomposition (1.1), then $\kappa^2_m(\delta) = \sum \kappa_m(\alpha_i) \otimes \kappa_m(\beta^i)$. In particular, Theorem 1.1 implies that

$$\sum \kappa_m(\epsilon_i) \otimes \kappa_m\bigl(\eta^i\bigr) = [\Delta_m]. \tag{5.3}$$

It follows from the definition of l that

$$l_m(a) = \sum_i \Bigl(\int_{Y_m} a \cup \kappa_m(\epsilon_i) \Bigr) \eta^i.$$

Applying κ_m to both sides, taking into account (5.3) and using 5.1, we compute:

$$\kappa_m l_m(a) = \sum_i \Bigl(\int_{Y_m} a \cup \kappa_m(\epsilon_i) \Bigr) \kappa_m\bigl(\eta^i\bigr) = a.$$

Hence Corollary 1.2 is proved.

5.3. Proof of Theorem 1.4

We first prove that $\kappa'^2_m[\Delta_{\mathbb{C}^*}] = [\Delta'_m]$, where $[\Delta'_m]$ is the Poincaré dual of the diagonal class in Y'_m. For that it suffices to check that the vector field ∇h is transverse to $h'^{-1}(m)$ and to apply the same arguments as in Section 5.1. The definition of h' given in [9] depends on some choices (which do not affect the map κ_m) and we will prove the required transversality when h' is a small enough perturbation of h (we might need a smaller perturbation than [9]). Away from a neighborhood of the fixed point set in $h^{-1}(m)$ the function h' coincides with h, so its m-level set is transverse to ∇h. So it suffices to look at a neighborhood of some fixed point component $Y \subset h^{-1}(m)$.

Let $N \to Y$ be the normal bundle of the inclusion $Y \subset X$, with its induced complex hermitian structure. The action of S^1 induces a splitting in complex subbundles $N = V^+ \oplus V^- = (V_1^+ \oplus \cdots \oplus V_s^+) \oplus (V_1^- \oplus \cdots \oplus V_k^-)$ and S^1 acts on $V_i^\pm$ with weight $\pm\lambda_i^\pm$ for some positive integers

$\lambda_1^+, \dots, \lambda_1^-, \dots$. There are neighborhoods $U \subset X$ of Y and $U_N \subset N$ of the zero section of N and a diffeomorphism $f: U_N \to U$ such that for any $(v^+, v^-) = (v_1^+, \dots, v_s^+, v_1^-, \dots, v_k^-) \in U_N$ we have $h \circ f(v^+, v^-) = m + \|v^+\|^2 - \|v^-\|^2$, where $\|v^+\|^2 = \sum_i \lambda_i^+ \|v_i^+\|^2$ and $\|v^-\|^2 = \sum_j \lambda_j^- \|v_j^-\|^2$, and also

$$\nabla h \circ f(v^+, v^-) = 2(\lambda_1^+ v_1^+, \dots, \lambda_s^+ v_s^+, -\lambda_1^- v_1^-, \dots, -\lambda_s^- v_k^-) + O(\|v^+\|^2 + \|v^-\|^2).$$

Assume that for some $\delta > 0$ the set $\{(v^+, v^-) \in N \mid \|v^+\|^2 + \|v^-\|^2 < 3\delta\}$ is contained in U_N. Let $\rho: \mathbb{R} \to \mathbb{R}$ be a smooth function with $\rho' \leqslant 0$, $\rho(t) = 1$ for $t < \delta$ and $\rho(t) = 0$ for $t > 2\delta$. Take $\epsilon \in \mathbb{R} \setminus \{0\}$ with $|\epsilon| < \min\{\sup |\rho'(t)|, \delta\}$, and require ϵ to be positive if and only if $\operatorname{rk} V^+ \leqslant \operatorname{rk} V^-$. Then the restriction of h' to U is defined as $m + \phi \circ f^{-1}$, where

$$\phi(v^+, v^-) = \|v^+\|^2 - \|v^-\|^2 + \epsilon\rho(\|v^+\|^2 + \|v^-\|^2).$$

Hence we need to prove that if (v^+, v^-) satisfies $\|v^+\|^2 + \|v^-\|^2 < 3\delta$ and $\phi(v^+, v^-) = 0$, then $d\phi(v^+, v^-)(\nabla h \circ f) > 0$. Using $\phi(v^+, v^-) = 0$ one computes

$$\begin{aligned}\frac{1}{2} d\phi(v^+, v^-)(\nabla h \circ f) = {} & \left(\sum_i (\lambda_i^+)^2 \|v_i^+\|^2 + \sum_j (\lambda_j^-)^2 \|v_j^-\|^2\right) \\ & + \epsilon\rho'(v^+, v^-)\left(\sum_i (\lambda_i^+)^2 \|v_i^+\|^2 - \sum_j (\lambda_j^-)^2 \|v_j^-\|^2\right) \\ & + O(\|v^+\|^3 + \|v^-\|^3).\end{aligned}$$

The first expressions in big parenthesis is not less than the second one, and nonzero if $\phi(v^+, v^-) = 0$, so if the O term was absent then we would have $d\phi(v^+, v^-)(\nabla h \circ f) > 0$. Picking δ small enough and $|\epsilon| \leqslant \sup |\rho'|/2$, the O term will be smaller than the first two terms, so the gradient will still be > 0. This finishes the proof that $\kappa_m'^2[\Delta_{\mathbb{C}^*}] = [\Delta_m']$.

Arguing as in the proof of Corollary 1.2 we deduce from $\kappa_m'^2[\Delta_{\mathbb{C}^*}] = [\Delta_m']$ that the element $(PD' \otimes \mathrm{Id}) \circ (\kappa_m' \otimes \mathrm{Id})[\Delta_{\mathbb{C}^*}]$ (where $PD': H^*(Y_m') \to H^{2n-2-*}(Y_m')^*$ denotes the Poincaré duality map) corresponds to a map $l_m': H^*(Y_m') \to H_{S^1}^*(X)$ which is a right inverse of κ_m'. By an argument in linear algebra $(f_H^* \otimes \mathrm{Id}) \circ (PD' \otimes \mathrm{Id}) \circ (\kappa_m' \otimes \mathrm{Id})[\Delta_{\mathbb{C}^*}]$ corresponds to $l_m := l_m' \circ f_H: IH^*(Y_m') \to H_{S^1}^*(X)$, which is a right inverse of $\kappa_m = f_H^{-1} \circ \kappa_m'$. Recall that $PD: IH^*(Y_m) \to IH^{2n-2-*}(Y_m)^*$ denotes the Poincaré duality map. Since f_H preserves the intersection pairing we have $f_H^* \circ PD' = PD \circ f_H^{-1}$, so

$$\begin{aligned}(f_H^* \otimes \mathrm{Id}) \circ (PD' \otimes \mathrm{Id}) \circ (\kappa_m' \otimes \mathrm{Id})[\Delta_{\mathbb{C}^*}] &= (PD \otimes \mathrm{Id}) \circ (f_H^{-1} \otimes \mathrm{Id}) \circ (\kappa_m' \otimes \mathrm{Id})[\Delta_{\mathbb{C}^*}] \\ &= (PD \otimes \mathrm{Id}) \circ (\kappa_m \otimes \mathrm{Id})[\Delta_{\mathbb{C}^*}].\end{aligned}$$

This proves the theorem.

5.4. An example

Let $\gamma : \mathbb{R} \to X$ be a gradient flow line of h such that

$$\lim_{t\to-\infty} h\big(\gamma(t)\big) = \sup h \quad \text{and} \quad \lim_{t\to\infty} h\big(\gamma(t)\big) = \inf h.$$

Then $E = S^1 \cdot \gamma(\mathbb{R}) \subset X$ is an S^1-invariant 2-dimensional sphere embedded in X, which defines an equivariant cohomology class $\alpha \in H^{2n-2}_{S^1}(X)$ (for example, taking any finite-dimensional approximation $X_\Lambda = S_\Lambda \times_{S^1} X$, we define $\alpha_\Lambda \in H^{2n-2}(X_\Lambda)$ as the Poincaré dual of $S_\Lambda \times_{S^1} E$; making then Λ go to ∞, the classes α_Λ define a unique class α). The class α is independent of the choice of γ.

One can prove that for any regular value $m \in \mathbf{i}\mathbb{R}$ we have $l_m(PD[\mathrm{pt}]) = \alpha$. This implies in particular that if $m' \in \mathbf{i}\mathbb{R}$ is another regular value then the map

$$\kappa_{m'} \circ l_m : H^{2n-2}(Y_m) \to H^{2n-2}(Y_{m'})$$

is an isomorphism of vector spaces. An immediate consequence is that l_m is not in general a morphism of rings. Indeed, if l_m were a morphism of rings then $\kappa_{m'} \circ l_m$ would also be a morphism of rings. But one can construct examples in which Y_m is the blow up of $Y_{m'}$ at a point and in this case, denoting by $\epsilon \in H^2(Y_m)$ the Poincaré dual of the exceptional divisor, it can be checked that $\kappa_{m'} \circ l_m(\epsilon) = 0$. However, $\epsilon^{n-1} \neq 0$, so $\kappa_{m'} \circ l_m(\epsilon^{n-1}) \neq 0$.

6. Actions of compact tori of arbitrary dimension

We now sketch how to generalize the previous constructions in order to prove Theorem 1.5. Fix a subgroup $S^1 \simeq T_0 \subset T$ such that the T_0-fixed point set coincides with the T-fixed point set and take a basis $\mathbf{u} = \{u_1, \dots, u_q\}$ of $\mathfrak{t}$. For each l let $\mathcal{X}_l$ denote the vector field generated by the infinitesimal action of u_l. The hypothesis on T_0 implies that for any two connected components F', F'' of the zero set of $\mathcal{X}_{u_l}$ satisfying $f' = \langle \mu(F'), u_l \rangle < f'' = \langle \mu(F''), u_l \rangle$ there is some $f' < a < f''$ such that the set $X_a = \{x \in X \mid \langle \mu(x), u_l \rangle = a\}$ does not contain any T_0-fixed point. Indeed, by hypothesis a T_0-fixed point is a zero of $\mathcal{X}_l$, and the set of values of the function $\langle \mu(\cdot), u_l \rangle$ evaluated at zeroes of $\mathcal{X}_l$ is finite.

Since T_0-stabilizers of the points in the level sets X_a are all finite, we can construct T_0-invariant multivalued perturbations of the equation $\gamma' = -I\mathcal{X}_l$ supported near the sets of the form X_a, just as we did in the case of actions of the circle. Thus we get a finite-dimensional space of perturbations $\mathbb{J}$ and, for any $1 \leqslant l \leqslant q$ and any $J \in \mathbb{J}$, a space $\mathcal{C}_{l,J}$ parameterizing oriented chains of J-perturbed gradient segments of $\langle \mu, u_l \rangle$, as in Section 2.4. We can also define, generalizing Section 3.3, the spaces

$$\mathcal{C}_J = \big\{(\mathbf{K}_1, \dots, \mathbf{K}_q) \in \mathcal{C}_{1,J} \times \cdots \times \mathcal{C}_{q,J} \bigm| b(\mathbf{K}_{i+1}) = e(\mathbf{K}_i)\big\},$$

$\mathcal{P}_J = T \times \mathcal{C}_J$, the maps $(b, e) : \mathcal{C}_J \to X \times X$ sending $(\mathbf{K}_1, \dots, \mathbf{K}_q)$ to $(b(\mathbf{K}_1), e(\mathbf{K}_q))$ and $\Theta_J : \mathcal{P}_J \to X \times X$ sending $(\theta, \mathbf{K}_1, \dots, \mathbf{K}_q)$ to $(\theta \cdot b(\mathbf{K}_1), e(\mathbf{K}_q))$, and the action of $T \times T$ on $\mathcal{P}_J$ defined as

$$(\alpha, \beta) \cdot (\theta, \mathbf{K}_1, \dots, \mathbf{K}_q) = \big(\alpha\beta^{-1}\theta, \beta \cdot \mathbf{K}_1, \dots, \beta \cdot \mathbf{K}_q\big).$$

Define also for any natural $r \geqslant 0$ the set $\mathcal{C}_J^r$ as the union, over all tuples $r_1, \dots, r_q$ of non-negative integers adding r, of the sets $\mathcal{C}_J \cap (\mathcal{C}_{1,J}^{r_1} \times \dots \times \mathcal{C}_{q,J}^{r_q})$. Finally, let $\mathcal{C}_J^{0,0} = \mathcal{C}_J \cap (\mathcal{C}_{1,J}^{0,0} \times \dots \times \mathcal{C}_{q,J}^{0,0})$ and define the weight of $(\mathbf{K}_1, \dots, \mathbf{K}_q) \in \mathcal{C}_J^{0,0}$ to be the product of weights $\mathrm{weight}(\mathbf{K}_1) \dots \mathrm{weight}(\mathbf{K}_q)$.

As in the case of S^1 there are finite-dimensional approximations of the universal bundle $ET \times ET \to BT \times BT$ of the form $E_\Lambda \to B_\Lambda$ (which are the qth Cartesian product of the corresponding fibrations for S^1), for any natural number Λ, and we can consider the fiberwise product $\mathcal{P}_{J,\Lambda} = E_\Lambda \times_{T \times T} \mathcal{P}_J$ and the map $\Phi_{J,\Lambda} : \mathcal{P}_{J,\Lambda} \to X_\Lambda^2$.

Theorems 3.1 and 3.3 generalize straightforwardly to the present situation with a few modifications which we now explain. The parametrization of neighborhoods of $\mathcal{C}_J$ given by Theorem 3.1 will be given by manifolds of dimension $2n + q$. Similarly one should modify the dimensions given in Lemmata 4.1 and 4.2 by adding $q - 1$ in each case. Finally, Lemma 4.3 should be generalized as follows. Let $T_1 \subset T$ be a subgroup such that $T = T_0 \times T_1$. Then the omega limit set of the map $\Phi_{J,\Lambda} : \mathcal{P}_{J,\Lambda} \to X_\Lambda^2$ can be covered by the images of smooth maps with domains of the form $\mathcal{Z}_i \times T_1 \times \mathcal{C}_J^1$, where $\mathcal{Z}_i$ has the same meaning as in Lemma 4.3. This still implies that the omega limit set has codimension $\geqslant 2$ and hence allows to prove Theorem 3.3.

As in Section 3.4 one can make Λ go to ∞ and obtain a cohomology class

$$\left[\Delta_{\mathbb{C}}^{T_0,T}\right] \in H_{T \times T}^{2n-2k}(X \times X).$$

To check that this class is independent of the basis $\mathbf{u}$ note: (1) if we replace $\{u_1, \dots, u_k\}$ by $\{-u_1, \dots, u_k\}$ then we get trivially the same cohomology class and (2) two different basis are homotopic up to reversing orientation. By an easy deformation argument similar to (4) in Theorem 3.3 we deduce the independence on $\mathbf{u}$.

Finally, the same arguments as in the proof of Theorem 1.1 (see Section 5) allow to prove that $\kappa_m^2([\Delta_{\mathbb{C}^*}^{T_0,T}]) = [\Delta_m]$.

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